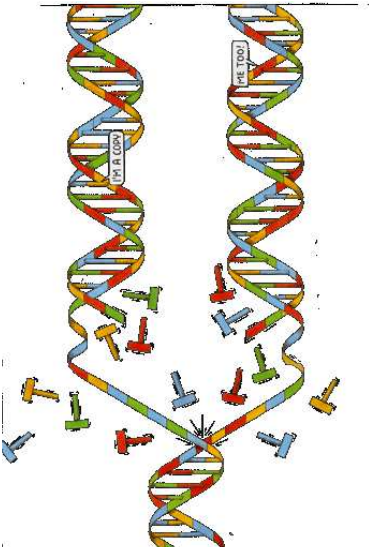
Problems for classical GT

- multiple equilibria $\Rightarrow$ no predictions possible
- "perfectly rational player" is too strong an idealization

Evolutionary Game Theory

- populations of players
- individuals are (genetically) programmed for certain strategy
- individuals replicate and thereby pass on their strategy



Utility and fitness

- number of offspring is monotonically related to average utility of a player
- high utility in a competition means the outcome improves reproductive chances (and vice versa)
- number of expected offspring (Darwinian "fitness") corresponds to **expected utility** against a population of other players
- genes of individuals with high utility will spread

Extinction of non-rationalizable strategies

- strictly dominated strategies always have less-than-average reproduction rate
- their proportion thus converges towards zero
- once a strictly dominated strategies dies out (or almost dies out), it can be ignored in the utility matrix
- corresponds to *elimination of a strictly dominated strategy*
- process gets iterated in evolutionary dynamics
- long-term effect:

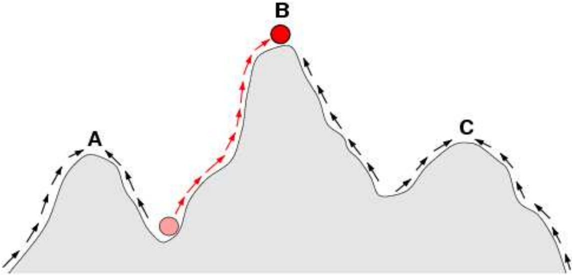
Theorem

If a strategy a_i is iteratively strictly dominated, then

$$\lim_{t \to \infty} p_t(a_i) = 0$$

Evolutionary stability

- Darwinian evolution predicts ascent towards local fitness maximum
- once local maximum is reached: stability
- only random events (genetic drift, external forces) can destroy stability
- central question for evolutionary model: what are stable states?



Evolutionary stability (cont.)

- replication sometimes unfaithful (mutation)
- population is **evolutionarily stable** $\rightsquigarrow$ resistant against small amounts of mutation
- Maynard Smith (1982): static characterization of

  Evolutionarily Stable Strategies

  (ESS) in terms of utilities only
- related to Nash equilibria, but slightly different

Evolutionary stability (cont.)

Rock-Paper-Scissor

	R	P	S
R	0	-1	1
P	1	0	-1
S	-1	1	0

- one symmetric Nash equilibrium: $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$
- not evolutionarily stable though

Evolutionary stability (cont.)

Pigeon orientation game

- "players" are pigeons that go together on a journey
- A-pigeons can find their way back, B-pigeons cannot

	A	B
A	1	1
B	1	0

Evolutionary stability (cont.)

- A is a non-strict Nash equilibrium, but nevertheless evolutionarily stable
- to be evolutionarily stable, a population must be able either
    - to fight off invaders directly (strict Nash equilibrium)
    - to successfully invade the invaders (non-strict Nash equilibrium)

Evolutionary Stable Strategy

Definition

The mixed strategy α is an **Evolutionarily Stable Strategy** *in a symmetric two-person game iff*

- $U(\alpha, \alpha) \geq U(\alpha', \alpha)$ *for all* α, *and*
- *if* $U(\alpha, \alpha) = U(\alpha', \alpha)$ *for some* $\alpha' \neq \alpha$, *then* $U(\alpha, \alpha') > U(\alpha', \alpha')$.

Strict Nash Equilibria

$\subset$

Evolutionarily Stable Strategies

$\subset$

Nash Equilibria

The Replicator Dynamics

- implicit assumption behind notion of ESS
  - Populations are (practically) infinite.
  - Each pair of individuals is equally likely to interact.
  - The expected number of offspring of an individual (i.e., its fitness in the Darwinian sense) is monotonically related to its average utility.
- can be made explicit in a dynamic model

Replicator Dynamics (cont.)

- easiest correlation between utility and fitness:

$$u(i,j) \quad = \quad \begin{array}{l} \textit{expected number of offspring} \\ \textit{of an individual of type } i \\ \textit{in a } j\textit{-population} \end{array}$$

Replicator Dynamics (cont.)

Suppose

- time is discrete
- in each round, each pair of players is equally likely to interact

Replicator Dynamics (cont.)

Discrete time dynamics:

$$N_i(t+1) = N_i(t) + N_i(t)(\sum_{j=1}^{n} x_j u(i,j) - d)$$

$N(t)$ ... population size at time t
$N_i(t)$ ... number of players playing strategy s_i
$x_j(t)$ ... $\frac{N_j(t)}{N(t)}$
d ... death rate

Replicator Dynamics (cont.)

generalizing to continuous time:

$$N_i(t + \Delta t) = N_i + \Delta t N_i(\sum_{j=1}^{n} x_j u(i,j) - d)$$

thus

$$\frac{\Delta N_i}{\Delta t} = N_i(\sum_{j=1}^{n} x_j u(i,j) - d)$$

Replicator Dynamics (cont.)

if $\Delta t \to 0$

$$\frac{dN_i}{dt} = N_i(\sum_{j=1}^{n} x_j u(i,j) - d)$$

Replicator Dynamics (cont.)

size of entire population may also change:

$$\begin{aligned} N(t+\Delta t) &= \sum_{i=1}^{n}(N_i + \Delta t(N_i \sum_{j=1}^{n} x_j u(i,j) - d)) \\ &= N + \Delta t(N \sum_{i=1}^{n} x_i \sum_{j=1}^{n} x_j u(i,j)) \end{aligned}$$

hence

$$\frac{dN}{dt} = N(\sum_{i=1}^{n} x_i (\sum_{j=1}^{n} x_j u(i,j) - d))$$

Replicator Dynamics (cont.)

let

$$\sum_{j=1}^{n} x_j u(i,j) = \tilde{u}_i$$

$$\sum_{i=1}^{n} x_i \tilde{u}_i = \tilde{u}$$

then we have

$$\frac{dN_i}{dt} = N_i(\tilde{u}_i - d)$$

$$\frac{dN}{dt} = N(\tilde{u} - d)$$

Replicator dynamics (cont.)

remember some calculus?

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

Replicator dynamics (cont.)

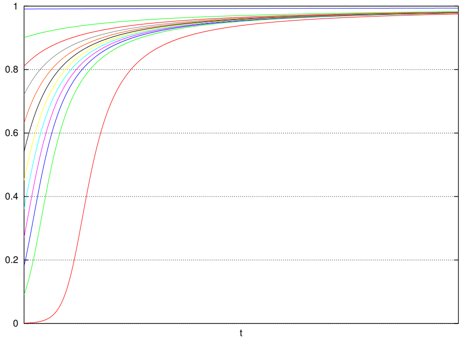
remember some calculus?

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

$$\begin{aligned}\frac{dx_i}{dt} &= \frac{(NN_i(\tilde{u}_i - d) - (N_i N(\tilde{u}_i - d)))}{N^2} \\ &= x_i(\tilde{u}_i - \tilde{u})\end{aligned}$$

Pigeon orientation

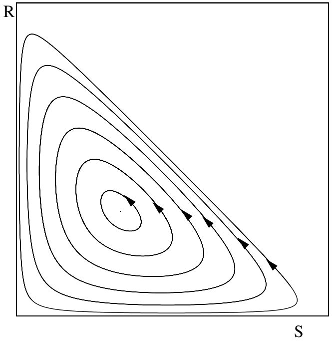
- ESSs correspond to **asymptotically stable states**
- a.k.a. **attractors**
- sample dynamics:



x-axis: time
y-axis: proportion of A-players

Rock-Paper-Scissor again

- three-strategy game: two independent variables
  - number of R-players
  - number of P-players
- number of S-players follows because everything sums up to 1
- supressing time dimension gives **orbits**



Asymmetric games

- symmetric games:
  - same strategy set for both players
  - $u_A(i, j) = u_B(j, i)$ for all strategies s_i, s_j
  - evolutionary interpretation: symmetric interaction *within one population*
- asymmetric games:
  - players have different strategy sets or utility matrices
  - evolutionary interpretation
    - different roles within one population (like incumbent vs. intruder, speaker vs. hearer, ...), or
    - interaction between disjoint populations
- evolutionary behavior differs significantly!

Asymmetric games (cont.)

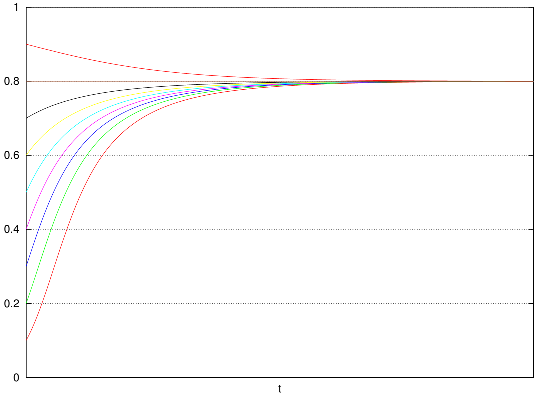
Hawks and Doves

	H	D
H	1,1	7,2
D	2,7	3,3

- can be interpreted symmetrically or asymmetrically
- symmetric interpretation:
  - hawks prefer to interact with doves and vice versa
  - ESS: 80% hawks / 20% doves
  - both strategies have average utility of 2.2
  - dynamics:

Symmetric Hawk-and-doves

- if hawks exceed 80%, doves thrives, and vice versa
- 80:20 ratio is only attractor state

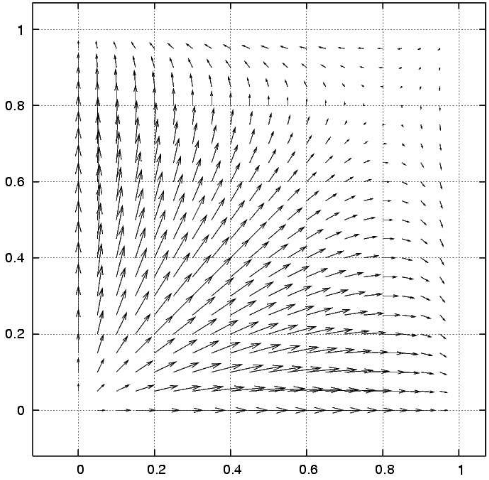


Asymmetric Hawks-and-doves

- suppose two-population setting:
  - both A and B come in hawkish and dovish variant
  - everybody only interacts with individuals from opposite "species"
  - excess of A-hawks helps B-doves and vice versa
  - population push each other into opposite directions

Hawks and doves

- 80:20 ratio in both populations is stationary
- not an attractor, but repellor



Asymmetric stability

- crucial difference to symmetric games:
  mutants do not play against themselves
- makes second clause of the symmetric ESS superfluous

Theorem (Selten 1980)

In asymmetric games, a configuration is an ESS iff it is a strict Nash equilibrium.

Asymmetric replicator dynamic

$$\begin{aligned}
\frac{dx_i}{dt} &= x_i\left(\sum_{j=1}^{n} y_j u_A(i,j) - \sum_{k=1}^{n} x_k \sum_{j=1}^{n} y_j u_A(k,j)\right) \\
\frac{dy_i}{dt} &= y_i\left(\sum_{j=1}^{m} x_j u_B(i,j) - \sum_{k=1}^{n} y_k \sum_{j=1}^{m} x_j u_B(k,j)\right)
\end{aligned}$$

x_i ... proportion of s_i^A within the A-population
y_i ... proportion of s_i^B within the B-population

Symmetrizing asymmetric games

- asymmetric games can be "symmetrized"
- correspondig symmetric game shares Nash equilibria and ESSs
- new strategy set:

$$S^{AB} = S^A \times S^B$$

- new utility function

$$u^{AB}(\langle i,j\rangle, \langle k,l\rangle) = u^A(i,l) + u^B(j,k)$$

Exercises

1. Find the symmetric ESSs of the following games (provided they exist):
    - Prisoner's dilemma
    - Stag hunt
2. Find the asymmetric ESSs of the following games (again, provided they exist):
    - Bach or Stravinsky
    - Matching pennies
3. Symmetrize the asymmetric version of Hawks and Doves and find the symmetric ESSs of the result. Which configuration in the original game do they correspond to?