Lambda-Terms with Functional Symbols and Decidability in Certain Closed Categories
S. V. Soloviev 309

Tutorial on Linear Logic
A. S. Troelstra 327

The Landscape of Deduction
J. van Benthem 357

Index 377

A Historical Introduction to Substructural Logics

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The common denominator of several important nonclassical logics is that in their sequent formulation they reject or restrict some of Gentzen's structural rules, whereas rules for logical constants need not differ from rules that may be given for classical logic. It is as if the structural part of logic were more fundamental: to change logic, we have to change this part. Logical constants are in principle secondary: they are invariant, they play the same role in different structural contexts.

We shall call logics that can be obtained in this manner, by restricting structural rules, *substructural* logics. This term may convey that something has been subtracted in the structural part of logic, but it should also convey, according to the standard meaning of the word 'substructural', that the restricted logic is the product of interventions in the supporting structures of logic.

The most important substructural logic is intuitionistic logic. That it is such a logic was discovered by Gentzen. Other important substructural logics investigated up to now, which make the subject of this introduction and of the rest of this book, are relevant logic, BCK logic, linear logic and the Lambek calculus of syntactic categories. These logics have various inspirations. Intuitionistic logic is a product of the constructivist side in the great debate on the foundations of mathematics. Relevant logic (though its earliest roots are mixed with the roots of intuitionistic logic; see §3 below) has a more philosophical inspiration: it grew out of attempts to formalize an implication devoid of the unintuitive features of material implication. The first reason for constructing the older contractionless logic, called BCK logic, was to evade set-theoretical paradoxes while keeping unrestricted comprehension (systems with restricted contraction have also been studied in many-valued logic and in connexion with extensions of intuitionistic logic; see §4 below). The younger contractionless logic, linear logic, draws its inspiration, and much of the attention it has attracted, from computer science. Finally, the Lambek calculus belongs primarily to mathematical linguistics, but it has also deep connexions with category theory. In the production of substructural logics, there has also been, no doubt, a certain amount of experimentation, of combinatorial playing, with the possibilities of the apparatus that makes the base of Gentzen's formalism (this is especially clear in some papers on BCK logic; see §4 below). There are infinitely many possibilities here, but we concentrate only on those that have proved their worth through prolonged serious investigation.

The substructural logics of this book do not differ only in inspiration: they are also separated by time and space. They have been discovered and rediscovered during the last sixty odd years by logicians in places wide apart, who would often not know about the work of the others. In spite of a great deal of overlapping in their results, their styles are sometimes so different as to prevent understanding. This book is produced in the hope that separation and ignorance may end and a united field of substructural logics may emerge.

The present introduction is about the earlier history of substructural logics. For newer developments and current research, one may consult the references listed at the end, or papers that follow in this book. In such a brief review, I can hardly do much more than mention those earlier works I know about and find worth mentioning. I shall dwell longer on some less well known, less cited, less accessible, papers than on some better known papers that are bound to receive their due.

1 Structural Rules

A very important discovery in Gentzen's thesis [1935] is that in logic there are rules of inference that don't involve any logical constant. Gentzen called such rules *structural*. The discovery of structural rules should not be ascribed uniquely to Gentzen but also to Hertz, who influenced Gentzen (see [Hertz 1922, 1928, 1929] and Gentzen's first paper [1932]). A prefiguration of structural rules may also be found in the axiom system for logic that Hilbert presented often in the twenties (see below). Of course, structural rules are so fundamental to logic that it must be possible to find an implicit conception of them from the very beginning of logic. However, a full and clear conception appears first in [Gentzen 1935]. Let us now see how Gentzen formulated these rules.

Let L be a language for whose formulae we use the schematic letters $A, B, C, \ldots$; for sequences of formulae of L, let us use the schematic letters $X, Y, Z, \ldots$. A *sequent* is, as in [Gentzen 1935], an expression of the form $X \vdash Y$. If Y consists of a single formula or is empty, $X \vdash Y$ is a

single-conclusion sequent; otherwise, it is a *multiple-conclusion* sequent. If we interpret a single-conclusion sequent $X \vdash C$ as asserting that there is an inference from the premises in X to the conclusion C, ordinary rules of inference may be formulated by sequent-schemata. For example, *modus ponens* may be formulated by the sequent-schema $A, A \rightarrow B \vdash B$. Gentzen's structural rules are different from these ordinary rules because their instances have inferences as premises and conclusions. We may say that ordinary rules are of *first level*, whereas Gentzen's structural rules are of *second level*. In [Gentzen 1935] we find the prototypes of the following structural rules, appropriate for a single-conclusion sequent system:

Thinning $\dfrac{X, Y \vdash Z}{X, A, Y \vdash Z}$

Contraction $\dfrac{X, A, A, Y \vdash Z}{X, A, Y \vdash Z}$

Permutation $\dfrac{X, A, B, Y \vdash Z}{X, B, A, Y \vdash Z}$

Cut $\dfrac{X \vdash A \qquad U, A, Y \vdash Z}{U, X, Y \vdash Z}$

Thinning may already be found in Hertz's papers mentioned above (Hertz calls it *unmittelbarer Schluß*, i.e. immediate inference). The term *Thinning* as a translation of Gentzen's *Verdünnung* may be found in Kleene's book [1952, §77], whereas the alternative translation *Weakening* may be found in Curry's books [1950, chapter II.5] and [1963, chapter 5C]. The idea is that $X, A, Y \vdash Z$ is a weaker, diluted, assertion when compared with $X, Y \vdash Z$. More recently, some authors have given the name *Monotonicity* to Thinning. This usage, probably derived from Scott's [1971] and some other of his papers from the same period, is justified by the equivalence of Thinning with the monotonicity of Tarski's consequence operation:

$$\textit{if } \mathcal{A} \subseteq \mathcal{B}, \textit{ then } \mathrm{Cn}(\mathcal{A}) \subseteq \mathrm{Cn}(\mathcal{B})$$

(cf. [Tarski 1930, 1930a]). It has become entrenched in the area of nonmonotonic logics (however, these logics may lack other structural rules besides unrestricted Thinning; see the end of §3 below).

The term *Contraction* seems to be universally accepted, whereas an alternative to the term *Permutation*, found in Curry's works, is Kleene's *Interchange* (some authors say *Exchange*). So, our *Thinning* and *Permutation* is a compromise between Kleene's and Curry's usage (the translation in [Gentzen 1969] follows Kleene's usage). The rule Cut, which Gentzen introduced in [1932], was prefigured by Hertz. Those who call Thinning

Monotonicity tend to call Cut *Transitivity* (and Identity below they would call *Reflexivity*).

A prefiguration of structural rules may be found in an axiom system of Hilbert for intuitionistic implication, conjunction and disjunction, which Hilbert and Bernays in [1934, chapter III, §3, p. 67] and [1939, supplement III, p. 442] call *positive logic*. By adding appropriate axioms for negation, Hilbert obtains classical logic from positive logic. The implicational axioms of positive logic in the version of [Hilbert 1922, 1923, 1928], which was often cited and has influenced Gentzen (as [Gentzen 1935, section V, §2] witnesses), are given with the following names and in the following order:

Introduction of an Assumption:	$A \to (B \to A)$
Omission of an Assumption:	$(A \to (A \to B)) \to (A \to B)$
Interchange of Assumptions:	$(A \to (B \to C)) \to (B \to (A \to C))$
Elimination of a Proposition:	$(B \to C) \to ((A \to B) \to (A \to C))$

(for the English names of these axiom-schemata, we follow the translation of [van Heijenoort 1967, p. 465]). The first axiom-schema corresponds to Thinning, the second to Contraction and the third to Permutation. The fourth corresponds to the structural rule called *Association*, which we shall consider in §6 below, but it is also related to Cut. Cut is also related to the rule of *modus ponens*, which Hilbert assumes in addition to the axiom-schemata above (we have insisted upon the order in which these axiom-schemata are presented in Hilbert's papers because this order is matched by the order in which Gentzen introduces structural rules in [Gentzen 1935, section III, §1.21]: Thinning, Contraction, Permutation, Cut). The axioms for conjunction and disjunction of Hilbert's axiomatization of positive logic are also closely related to Gentzen's rules for these connectives.

By a natural extension of Gentzen's usage, we could take the term *structural* to cover some first-level rules as well as the second-level rules above. For all structural rules it would be characteristic that they can be formulated so that no constant of the language L occurs in their formulation. A structural rule of first level is given by the sequent-schema:

$$\textit{Identity:} \quad A \vdash A$$

which is an axiom-schema in Gentzen's sequent systems. Another structural rule of first level, not treated in [Gentzen 1935], could be given by the sequent-schema:

$$\textit{Substitution:} \quad A \vdash A^x_a$$

where A^x_a stands for the formula obtained from A by replacing a variable x by an expression a of the same syntactic category as x. This rule may be formulated for every language L that has variables of the syntactic category

of x and a. However, in the context of Gentzen's sequent systems, the first-level rule of Substitution is incorrect and we should not assume it (in other contexts, where introduction of implication on the right, i.e. the deduction theorem, is restricted, it can make sense; cf. [Kleene 1952, §23, p. 101]). Instead, we could assume the analogous second-level rule:

$$\frac{\vdash A}{\vdash A^x_a}$$

No constant of L occurs either in the sequent-schema of Identity or in the sequent-schema or second-level rule of Substitution. So, these rules are indeed structural.

Gentzen's and our structural rules should not be confused with what is often called *structural rules* in the Polish logic literature. There, a rule is called structural if its instances are closed under substitution (cf. [Wójcicki 1988, §2.1.7]).

Thinning, Contraction and Permutation as formulated above enable us to transform only the left-hand side of a sequent. In addition to these rules, Gentzen formulates in [1935] analogous structural rules for transforming the right-hand side of a sequent, and his Cut is like the following multiple-conclusion form of Cut:

$$\frac{X \vdash V, A, W \qquad U, A, Y \vdash Z}{U, X, Y \vdash V, Z, W}$$

All these structural rules, on the left and on the right, and the multiple-conclusion form of Cut are assumed for Gentzen's multiple-conclusion sequent system for classical logic. Gentzen passes from this system to the single-conclusion sequent system for intuitionistic logic by restricting the language to single-conclusion sequents.

However, the same effect can be obtained in another manner. Sequent systems for classical and intuitionistic logic can be so formulated that passing from the system for classical logic to the system for intuitionistic logic consists in restricting the rule of Thinning on the right to:

$$\textit{Intuitionistic Thinning on the right} \quad \frac{X \vdash}{X \vdash A}$$

Everything else, and, in particular, assumptions about logical constants, will be the same in both systems. For example, for implication we assume the following Gentzen-style rules:

$$(\to L)\ \frac{X \vdash V, A, W \qquad U, B, Y \vdash Z}{U, X, A \to B, Y \vdash V, Z, W} \qquad (\to R)\ \frac{A, X \vdash V, B, W}{X \vdash V, A \to B, W}$$

We define $\neg A$ as $A \to \bot$ (as in [Gentzen 1935, section II, §5.2]) and for $\bot$ we assume:

$$(\bot L)\ \bot \vdash \qquad (\bot R)\ \frac{X \vdash}{X \vdash \bot}$$

If we reject even Intuitionistic Thinning on the right, we obtain a system for Kolmogorov's and Johansson's minimal intuitionistic logic (see §2 below). So, intuitionistic logic may be conceived as a logic obtained by restricting the structural rules of classical logic.

Our proposal is to call logics that can be obtained in this manner, by restricting structural rules, *substructural* logics. The logic whose structural rules we restrict in order to obtain substructural logics is in principle classical logic. A substructural logic obtained by restricting the structural rules of intuitionistic logic is ultimately substructural relative to classical logic. Canonically, we should assume the same rules for logical constants in the logic whose structural rules we restrict, i.e. classical logic, and in the resulting substructural logic (according to [D. 1989], all alternative logics should arise by changing only the structural component of logic; assumptions about logical constants should be invariant). However, in practice, the situation is not always so clear-cut, either because a weaker structural context may make classically equivalent constants split into nonequivalent constants (see §3 below) or because we may consider various extensions of our logics (for example, modal extensions), which are not standard in classical logic. Sometimes, rules for logical constants may carry hidden structural assumptions, as rules appropriate for classical negation may transform intuitionistic logic into classical without explicit interventions concerning Thinning on the right. Such rules are inappropriate for formulating substructural logics that reject the hidden structural assumptions.

We don't insist that the Gentzen formulation of classical logic whose structural rules we restrict in order to obtain a substructural logic should be the standard one. It could as well be a nonstandard formulation with sequents whose left-hand and right-hand sides are not sequences of formulae of L but some other sort of structure involving formulae of L, like sets, or multisets, or trees, or sets of sets, or sets of multisets etc. (cf. the sequents of §6 below introduced for the nonassociative Lambek calculus; Dunn's sequents of [Anderson and Belnap 1975, §28.5] are related to sequents based on sets of multisets of sets etc.).

2 The First Axiomatizations of Intuitionistic Logic

The oldest substructural logics are older than [Gentzen 1935]. Intuitionistic logic was already axiomatized when Gentzen was writing his thesis. In [1925], Kolmogorov made the first attempt to axiomatize this logic by taking from [Hilbert 1923] Hilbert's axioms for implicational positive logic mentioned above and supplementing them with a *minimal*, i.e. purely implicational, intuitionistic negation (minimal negation $\neg A$ may be defined as $A \to \bot$ without any postulate being assumed about $\bot$, as in [Johansson 1936]). Kolmogorov's paper is interesting for us here because, in addition to rejecting $A \to (\neg A \to B)$, which corresponds to Intuitionistic Thinning on the right, he also considers briefly whether $A \to (B \to A)$, which corresponds to Thinning on the left, should be kept in intuitionistic logic (see [1925, section II, §4]). He decides that it should.

Glivenko hesitated in accepting $A \to (B \to A)$ and $A \to (\neg A \to B)$ before being convinced by Heyting that they should be taken into the axiomatization of intuitionistic propositional logic he produced in [1929]. Both of these principles appear as axioms in Heyting's axiomatization of [1930] (produced in 1928). Glivenko's axiomatization is built on a version of Hilbert's axioms for positive logic (Glivenko does not refer to Hilbert's positive logic or to Kolmogorov). In addition to Hilbert's first three implicational axioms, he has $A \to A$, which corresponds to the first-level structural rule of Identity ($A \to A$ is not independent from the other postulates of positive logic), and $(A \to B) \to ((B \to C) \to (A \to C))$ instead of $(B \to C) \to ((A \to B) \to (A \to C))$ (Hilbert also made this last change in [1931] and [Hilbert and Bernays 1934, chapter III, §3, p. 65], which makes Interchange of Assumptions dependent on the other postulates). Glivenko's axioms for conjunction and disjunction are as in [Hilbert and Bernays 1934]. In [1935, section V, §2], Gentzen preferred to reproduce Glivenko's axioms for intuitionistic propositional logic rather than Heyting's, presumably because these axioms are easier to connect with his systems (Heyting's axiomatization is closer in spirit to the axiom system of *Principia Mathematica*, from which it was derived; see [Troelstra 1990]).

3 Relevant Logic

In relevant logic (also called *relevance* logic) we reject Thinning on both the left and right. So, we will lack both $A \to (B \to A)$ and $A \to (\neg A \to B)$. It seems that the canonical version of relevant logic should be included in intuitionistic logic, since Thinning on the right is rejected. However, whether in this way we will end up with something included in intuitionistic logic depends on the exact rules for logical constants we have started from.

In particular, rules for negation may be formulated in such a way that, even after rejecting Thinnning on the right, we will obtain a multiple-conclusion sequent system with a nonintuitionistic, *involutive*, negation (i.e., in addition to $A \to \neg\neg A$, we will have $\neg\neg A \to A$), which also satisfies De Morgan's laws. Historically, relevant logic with an involutive negation was dominant from the very beginning.

Relevant logic antedates [Gentzen 1935] and is as old as intuitionistic logic. Its first inception is in [Orlov 1928], which unfortunately was unknown to later authors that developed relevant logic. In his paper, Orlov axiomatized the implication-negation fragment of the best-known relevant logic, later named R. Negation in this system is involutive. Orlov's aim was to formalize intuitionistic ideas about logic by further adding to his system an S4-type necessity operator. If we know about Kolmogorov's and Glivenko's reticence concerning $A \to (B \to A)$ and $A \to (\neg A \to B)$, we may find it natural that an author from their milieu, working on the formalization of intuitionistic logic, should appear with a system that lacks both of these principles.

With Orlov's paper, we may illustrate something typical for substructural logics; namely, the splitting of some old connectives. The fewer structural rules there are, the more distinctions can be made, and connectives equivalent in classical logic can now be distinguished from each other (the same phenomenon occurs already with intuitionistic logic, where, for example, $A \to B$ is not equivalent to $\neg(A \wedge \neg B)$). With the definition:

$$A \cdot B \;=_{df}\; \neg(A \to \neg B)$$

Orlov obtains a connective that resembles conjunction, but is now distinguished from it. This connective, which later acquired the name *fusion* in relevant logic, is important because it imitates the comma on the left-hand side of sequents. The essential principle governing fusion and implication is:

$$A \cdot B \vdash C \quad \textit{iff} \quad B \vdash A \to C$$

which does not depend on having, as Orlov, an involutive negation. Algebraically, this means that the operation corresponding to implication is the *residual* of a multiplication corresponding to fusion; i.e., we have:

$$a \cdot b \leq c \quad \textit{iff} \quad b \leq a \to c.$$

With classical and intuitionistic logic, this multiplication corresponds to conjunction and is a meet operation; with weaker substructural logics, it may lack some of the meet properties. With relevant logic, we lack $a \cdot b \leq a$ and $a \cdot b \leq b$, because we lack Thinning.

If an intuitive reading is asked for fusion, which would be distinct from *and*, we may try making a connective from *with* (or *together with*). This should mirror the conjoining of premises fusion performs. Whereas, *A and B* always implies A, in relevant logic *A with B* need not imply A, since B may be irrelevant.

With an involutive negation, we may define, as Orlov did, the De Morgan dual of fusion, later named *fission* in relevant logic:

$$A + B \;=_{df}\; \neg A \to B.$$

Fission resembles disjunction, but is different from it. As fusion imitates the comma on the left-hand side of sequents, so fission imitates the comma on the right-hand side of sequents. Implication, fusion, fission and negation make the so-called *intensional* fragment of relevant logic; nowadays, a more popular term than *intensional* seems to be *multiplicative*, which we borrow from [Girard 1987]. Conjunction and disjunction, which algebraically behave like respectively finite meet and finite join in a lattice (this lattice need not be distributive; see below), were later added to the multiplicative fragment. They are called *extensional*, or in the more fashionable terminology of [Girard 1987], which we follow here, *additive* connectives. A similar splitting between multiplicative and additive occurs for the propositional constants, i.e. nullary connectives, $\top$ and $\bot$. Multiplicative $\top$ and $\bot$, symbolized in [Anderson and Belnap 1975, §27.12, pp. 342–343] by respectively t and f (in [Girard 1987], we have instead 1 and $\bot$) behave algebraically like unit elements with respect to fusion and fission multiplication; i.e., we have $\mathsf{t} \cdot a = a$ and $\mathsf{f} + a = a$. Additive $\top$ and $\bot$, symbolized in [Anderson and Belnap 1975, *loc. cit.*] by T and F (in [Girard 1987], we have $\top$ and 0), behave like the greatest and least element in the lattice that corresponds to conjunction and disjunction. On the other hand, only additive versions of the quantifiers may be found in the existing systems of substructural logics. A multiplicative universal quantifier would presumably be something like an infinite fusion, but I am not aware that the idea has been investigated more thoroughly.

The reason why relevant logic appeared before the other substructural logics is probably that Thinning stands apart from other structural rules. Contraction and Permutation just serve to reduce sequences of formulae to sets of formulae, and Cut is something we try to eliminate as a primitive rule, but our systems should nevertheless be closed under it (systems not closed under Cut appear in the literature, but they are rather unusual). In Tarski's axiomatization of the consequence operation, which is an operation on sets of formulae, no axiom corresponds to Contraction and Permutation, but the monotonicity axiom, mentioned in §1 above, corresponds to Thinning. In [1936, §5.244], Gentzen finds he must justify Thinning, whereas other structural rules are accepted without further ado. He says for Thinning that:

> at first, it may seem somewhat strange; yet, for example, if a proposition is true, we are forced to admit that in that case it also holds on the basis of an *arbitrary assumption* (if we were to stipulate that this may be asserted only in cases where a 'factual dependence' ['*tatsächliche Abhängigkeit*'] exists, considerable difficulties would arise because of the possibility of proofs in which only an *apparent* use of an assumption is made) [Gentzen 1969, p. 152].

The first authors after Orlov prepared to tackle the difficulties of Gentzen's '*tatsächliche Abhängigkeit*', i.e. relevance, were Moh Shaw-Kwei in [1950] and Church in [1951]. They axiomatized the implicational fragment of R guided by a deduction theorem for a notion of deduction from hypotheses where every hypothesis is used in obtaining the conclusion. Church's interest in relevant implication should be related to his predilection for the λ-I-calculus, where $\lambda x.t$ is not well-formed if x does not occur free in the term t (cf. [Curry and Craig 1953]). In [Church 1941], the λ-I-calculus is the main lambda calculus, whereas the ordinary lambda calculus, called λ-K-calculus, is secondary. Thinning, Contraction and Permutation are related respectively to the combinators K, W and C via the Curry-Howard, formulae-as-types, interpretation, which encodes natural-deduction proofs with typed lambda terms (cf. [Lambek and Scott 1986, part I, §§1-15] and [Girard et al. 1989, chapter 3]). Natural-deduction proofs in substructural logics lacking, for example, Thinning will be coded by typed lambda terms that exclude a closed term corresponding to K, and similarly with other structural rules and combinators.

The later history of relevant logic is tied mainly to the work of Ackermann, Anderson, Belnap, Dunn, Meyer, Routley, Urquhart and logicians writing in response to their work. With these authors, relevant logic acquired its name by being welded with earlier attempts to formalize the intuitive non-truth-functional implication based on a connexion of meaning (modal logic has in its roots a similar philosophical motivation). This may seem like a hopelessly unmathematical enterprise, like fighting against zero on the ground that zero is not intuitively a number (cf. [Curry 1963, chapter 5, §2]). However, the best results of relevant logic don't bear so much on an implication in tune with intuitions (as shown by Lewis' famous argument from [Lewis and Langford 1932, p. 250; cf. Read 1988, p. 31] that A and not A implies B, these intuitions may well be contradictory beyond remedy), but on an implication arising from deductions where the assumptions are really involved in getting the conclusion. It is, after all, considered to be a mathematical improvement if a superfluous assumption is removed from a theorem (though the theorem was not wrong before the removal and though the advance is often judged in aesthetical terms). A related subject, which might seem more mathematical, is functions that really depend on their arguments — something akin to Church's λ-I-calculus.

The literature produced by the authors of the preceding paragraph is so big and has so many developments that we cannot possibly do it justice in such a brief introduction. One may consult [Dunn 1986], a shorter introductory text, and the encyclopaedic [Anderson and Belnap 1975], which has recently acquired a second volume [Anderson, Belnap and Dunn 1992]. Let us mention, however, that to the uninformed reader it may not be immediately clear that relevant logic should be characterized, as it was here, by the rejection of Thinning. Such a characterization, which is clearly given in [Kripke 1959], is sometimes hidden under the complications of rather nonstandard Gentzen formulations of the mainstream relevant logics. These complications are mainly due to the fact that in the school of Anderson and Belnap it is considered that we must assume distribution of conjunction over disjunction, i.e.:

$$(C \wedge (A \vee B)) \quad \rightarrow \quad ((C \wedge A) \vee (C \wedge B))$$

(by $\wedge$ and $\vee$ we symbolize additive conjunction and disjunction). If we simply reject Thinning from classical or intuitionistic logic, we will lack this principle. Instead, fusion will distribute over disjunction; i.e., we shall have:

$$(C \cdot (A \vee B)) \quad \rightarrow \quad ((C \cdot A) \vee (C \cdot B))$$

(the same thing happens if we reject Contraction). Distribution of conjunction over disjunction is behind many technically unwholesome properties of relevant logic, such as undecidability of propositional logic and failure of interpolation (which were quite difficult to establish). But, most important to us here, the wish to assimilate it may hide the substructural nature of relevant logic. (The substructural logics without Contraction of §§4–5 below also lack distribution of conjunction over disjunction and have only distribution of fusion over disjunction.)

The central relevant logic from the substructural perspective should be a version of R minus distribution of conjunction over disjunction, with either an involutive or an intuitionistic purely implicational negation (this logic has recently gained some status among relevant logicians in [Thistlewaite et al. 1988], though it has been known from Meyer's thesis since the middle sixties; see also [Smirnov 1972, chapter 6] and [Smirnov 1973]). The relevant logics E and T, which in the sixties and seventies have received much attention in the Anderson and Belnap school, do not engender clear-cut substructural logics, as R does. An S4-style modal extension of R, prefigured already by Orlov in [1928], is a better candidate than E (but without $(\Box A \wedge \Box B) \rightarrow \Box(A \wedge B)$, which, though favoured in [Anderson and Belnap 1975, §27.1.3, pp. 343–344], is very much like distribution of conjunction over disjunction: in the absence of Thinning, it does not follow from standard S4-style sequent rules, but it follows when conjunction is replaced by fusion).

However, it is worth mentioning several logics in the neighbourhood, characterized by various restrictions on Thinning and possibly other structural rules. Such is the so-called *mingle* extension of relevant logic, with the following restricted version of Thinning, converse to Contraction:

$$\textit{Expansion}\quad \frac{X, A, Y \vdash Z}{X, A, A, Y \vdash Z}$$

(more information on mingle systems can be found in [Dunn 1986], [Anderson and Belnap 1975] and [Avron 1987]).

Somewhat farther from relevant logic, but still comparable to it, is modal logic based on strict implication. The strict implication $\rightarrow$ of S4 can be characterized by restricting Thinning and Permutation to:

$$\textit{S4 Thinning}\quad \frac{X, B \rightarrow C, Y \vdash Z}{X, A, B \rightarrow C, Y \vdash Z}$$

$$\textit{S4 Permutation}\quad \frac{X, B \rightarrow C, A, Y \vdash Z}{X, A, B \rightarrow C, Y \vdash Z}$$

In addition to these *quasi-structural* rules, we have Identity, unrestricted Contraction on the left, Cut, $(\rightarrow L)$ and $(\rightarrow R)$ from §1 above.

It seems that nonmonotonic logics should be connected with relevant logic, with which they share the rejection of unrestricted Thinning, but it also seems that, up to now, this connexion has not been much exploited (cf. [Thistlewaite et al. 1988, §1.2, pp. 9–11]). However, nonmonotonic logics may require some specific structural rules, like the following restricted versions of Thinning and Cut:

$$\frac{X, Y \vdash A \qquad X, Y \vdash B}{X, A, Y \vdash B}$$

$$\frac{X, Y \vdash A \qquad X, A, Y \vdash B}{X, Y \vdash B}$$

whereas unrestricted Thinning and Cut are rejected. Some additional Thinning would come through assuming $X, A, Y \vdash A$ rather than $A \vdash A$ (see [Gabbay 1985]). Note that this, together with the restricted Thinning above, yields Expansion as follows:

$$\frac{X, A, Y \vdash A \qquad X, A, Y \vdash B}{X, A, A, Y \vdash B}$$

With restrictions on structural rules such as we find in nonmonotonic logics, we are abandoning the combinatory inspiration that characterizes all the

main substructural logics. Namely, after rejecting a structural rule that corresponds to a well-known combinator, we are introducing something not inspired by such a combinator. Presumably, nonmonotonic logics would be better understood if methods developed for related substructural logics were applied in their investigation.

4 BCK Logic

In BCK logic we reject Contraction on both the left and right, and keep Thinning. Systems of many-valued logic (for example, the systems of Łukasiewicz) may lack Contraction, or have only restricted forms of Contraction. In this respect they resemble BCK logic. However, as a common distrust of Thinning does not suffice to conflate relevant logic with modal logic, so BCK logic is not to be conflated with many-valued logic. It is doubtful whether one should look for the roots of BCK logic in the works of the founders of many-valued logic, Vasil'ev, Łukasiewicz and Post.

We are nearer to the roots of BCK logic with a work contemporaneous with [Gentzen 1935]; namely, Fitch's doctoral thesis of 1934, from which [Fitch 1936] was adapted. Fitch's ambition at that time, as well as later, was to obtain a type-free system of logic in which paradoxes would not arise. However, at the propositional level, Fitch's system of [1936] is classical propositional logic. Contraction is rejected at the level of predicate logic, which is built in the manner of Schönfinkel and Curry, without type distinctions. Fitch's system, unlike ordinary predicate logic, does not have for every formula with two free occurrences of a variable an equivalent formula with a single free occurrence of this variable (in ordinary predicate logic, we have $A(x,x) \leftrightarrow \forall y(x = y \rightarrow A(y,y))$). This blocks the derivation of Russell's paradox. It is well-known from Curry's work that Contraction and the corresponding combinator are essentially involved in Russell's paradox (see [Curry and Feys 1958, chapters 5G, 8A-B, and p. 338]). A system with ambitions similar to Fitch's, but without Contraction and other structural rules at the propositional level, was later introduced in a series of papers by Ackermann starting with [1950].

In the middle thirties, the implicational fragment of BCK logic was considered by Tarski. He proved in [1935] that all implicational calculuses containing the implicational BCK calculus, which is obtained from Hilbert's implicational positive logic by leaving out $(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$, have the classical implicational calculus as their only consistent Post-complete extension closed under substitution for propositional variables. (To prove this result of Tarski, note that:

$$((A \rightarrow A) \rightarrow (A \rightarrow A)) \rightarrow (A \rightarrow A)$$

$$((A \rightarrow A) \rightarrow A) \rightarrow A$$

$$(A \to (A \to A)) \to (A \to A)$$

and the converse implications are provable in the implicational BCK calculus. If B is not a classical tautology, there must be a valuation v such that $v(B) = \bot$. For every propositional variable p in B, substitute $A \to A$ for p if $v(p) = \top$, and A if $v(p) = \bot$. It follows by replacement of equivalents that the extension of the implicational BCK calculus with B is not consistent. This proof shows that in Tarski's result we can replace the implicational BCK calculus by a weaker system.)

A system where Contraction is restricted to:

$$\frac{X, A, A, A, Y \vdash Z}{X, A, A, Y \vdash Z}$$

was investigated by Nelson in [1959], in connexion with intuitionistic logic extended with *constructible falsity*, i.e. an involutive negation that satisfies De Morgan's laws, which however does not make intuitionistic logic collapse into classical logic (see [Rasiowa 1974, chapters V and XII] and references therein; this negation resembles the negation of another type-free system of Fitch from [1952]). A similar system with restricted Contraction and involutive negation was investigated starting from the sixties by Zaslavskiĭ (these investigations are reported in his monograph [1978]). Zaslavskiĭ finds that his system is an intuitionistic analogue of Łukasiewicz's three-valued logic. The implication $\Rightarrow$ of this system may be defined by:

$$A \Rightarrow B \;\;=_{df}\;\; (A \to B) \wedge (\sim B \to \sim A)$$

where $\to$ is intuitionistic implication and $\sim$ is involutive negation (cf. the definition of $\Rightarrow$ in [Rasiowa 1974, pp. 68, 280]).

Next, we mention two papers from the sixties, dealing with implicational BCK logic. In [1963], Jaśkowski showed that a formula is provable in this logic iff it is obtained by identifying variables of a classical implicational tautology in which every variable occurs once or twice. In the second paper, [Meredith and Prior 1963], the name 'BCK' makes one of its first appearances (see also [Prior 1962, appendix I, p. 316]). This name is derived from the combinators corresponding to the axioms of implicational BCK logic according to the Curry-Howard interpretation (cf. §3 above).

In the middle sixties, Iséki started the investigation of algebras related to implicational BCK logic (see [Iséki and Tanaka 1978]). Apart from Japanese mathematicians, Wroński, his collaborators in Cracow and Bunder made the main contributions in this area (references are too numerous to be given here: I have counted more than twenty papers only in *Mathematica Japonica* for the first half of the eighties).

Grishin seems to be the first logician that investigated BCK logic extensively without restricting himself to the implicational fragment (a decidable sequent system of predicate logic without Contraction is investigated in [Smirnov 1972, chapter 5]; however, this system, which is not closed under Cut, does not amount to BCK logic). Grishin formulates BCK logic by rejecting Contraction from Gentzen's sequent system for classical predicate logic and obtains a system with an involutive negation that satisfies De Morgan's laws. Since we don't restrict Thinning, this, rather than an intuitionistic version, should be the canonical version of BCK logic (an intuitionistic version of BCK logic would restrict Thinning on the right). So, Grishin's approach is properly in the spirit of substructural logics. His initial motivation for considering BCK logic was similar to Fitch's, but he does not refer to Fitch, Curry or Ackermann. He sees himself as continuing the work of Bochvar, Skolem, Chang and Fenstad on set theory with unrestricted comprehension based on many-valued logic (similar investigations may be found in the area of paraconsistent logics; however, so far as I know, these investigations, though they resemble things in the relevant logic and BCK logic literature, don't seem to be in the spirit of substructural logics). In [1974], Grishin shows via cut elimination that unrestricted comprehension can be consistently added to his BCK predicate logic. He also concludes from cut elimination that BCK predicate logic, a first-order, possibly many-sorted, system, is decidable (a remark concerning this fact may be found in [Wang 1962, chapter IX, §1, p. 228]).

Moreover, Grishin introduces for every natural number n a translation t_n from BCK predicate logic into classical predicate logic, whose essential clause is:

$$t_n(\forall x A) \;\;=\;\; \forall x t_n(A) \cdot \ldots \cdot \forall x t_n(A)$$

where $\forall x A$ is a negative, i.e. antecedent, subformula and $\forall x t_n(A)$ is repeated n times in the fusion on the right-hand side (cf. [Ono 1990, §3]). He shows that A is provable in classical predicate logic iff, for some n, $t_n(A)$ is provable in BCK predicate logic. Since the latter system is decidable, the existential quantifier 'for some n' on the right-hand side of this equivalence is the culprit for the undecidability of classical predicate logic. Via the translation t_n, we see that this quantifier is tied to Contraction involving formulae of the form $\forall x A$ on the left-hand side of sequents. When, after having established cut elimination for classical predicate logic, we attempt to use it to show decidability, we shall run into unsurmountable difficulties with Contraction involving formulae of the form $\forall x A$ on the left-hand side (and, dually, with Contraction involving formulae of the form $\exists x A$ on the right-hand side).

In [1974], Grishin remarks that BCK logic is properly included in Łukasiewicz's infinite-valued logic, the latter not being recursively axiomatizable. In the same paper, a formulation of Łukasiewicz's three-valued logic is obtained by extending BCK logic with the following restricted version of Contraction on the left:

$$\frac{X, A+A, A+A, Y \vdash Z}{X, A+A, Y \vdash Z}$$

and the analogous quasi-structural rule on the right (the fission $A + A$ stands for $\neg A \rightarrow A$, as in §3 above). A similar, but more involved, restriction on Contraction, introduced at the end of [Grishin 1976], gives Łukasiewicz's four-valued logic. In subsequent papers [1976, 1979, 1981, 1983, 1985], Grishin studied the algebraic models of BCK logic, an analogue of Herbrand's theorem, and the effect of adding both unrestricted comprehension and extensionality to BCK logic. This effect is that we regain classical logic and the paradoxes (cf. [White 1979] for a similar result concerning Łukasiewicz's infinite-valued logic).

In [Dardžaniá 1977], results concerning decidability and translation similar to Grishin's are presented for an intuitionistic version of BCK predicate logic. A decision procedure for a fragment of Grishin's BCK predicate logic, whose propositional part is purely multiplicative, is described in [Ketonen and Weyhrauch 1984]. We shall not review in detail subsequent papers on BCK logic by Ono and Komori, of which the main is their joint paper [1985] (further references are in Ono's survey [1990]). These papers are devoted mainly to Kripke-style models for BCK and related substructural logics (see §7 below).

5 Linear Logic

In linear logic, we reject both Thinning and Contraction, on both sides. In this logic we have no structural rules except Identity and Cut if sequents are based on *multisets* of formulae instead of sequences of formulae as in [Gentzen 1935].

In a discussion of Church's idea of a minimal logic in [Curry 1954], one finds implicational systems obtained from the implicational fragment of the relevant logic R by restricting Contraction to:

$$\frac{X, A^{k+1}, Y \vdash Z}{X, A^k, Y \vdash Z}$$

where A^k stands for a sequence of k occurrences of A, for some $k \geq 2$. The suggestion to reject Contraction completely, as well as Thinning, to arrive at a minimal logic, may be found in [Kripke 1959]. An implication-negation calculus based on the same rejection is in [Smiley 1959, §5]; its implicational postulates are:

$$(B \rightarrow C) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$$

$$(A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))$$

$$A \rightarrow A$$

and *modus ponens*. In [Jaśkowski 1963], the theorems of this implicational calculus receive a characterization analogous to the characterization given in the same paper for implicational BCK logic (cf. [Smiley 1959, footnote 21]). This implicational calculus is called BCI in [Meredith and Prior 1963] and [Prior 1962, appendix I, p. 316], according to the combinators corresponding to the three implicational axiom-schemata above (cf. §4 above and [Meredith 1978]). Systems without Thinning and Contraction have also been considered in relevant and BCK quarters (see [Meyer and McRobbie 1982] and [Komori 1986]) and in the survey [D. 1988] (for additional references see [D. 1992a]).

Linear logic acquired its name in [Girard 1987] (according to Girard, there is an analogy between this logic and linear algebra). Girard is motivated by applications he foresees in computer science, but his project has also a more philosophical side expounded in [Girard 1989]. A novelty in [Girard 1987] is that besides the usual logical connectives, implication, negation, conjunction, disjunction (appearing in a rather colourful notation), and the less usual ones, fusion, fission and multiplicative and additive propositional constants, which were already considered in relevant and BCK logic, he has S4-like modal operators that serve to mimic the missing structural rules. Namely, we have restricted forms of Thinning and Contraction for modalized formulae. This makes it possible to embed intuitionistic and classical logic into linear logic by a modal translation (of course, analogous embeddings are available for other substructural logics, provided we add the requisite modal operators; see [D. 1992a]). However, linear propositional logic with the modal operators is undecidable (this is proved in [Lincoln et al. 1992]), though linear predicate logic without the modal operators is decidable, as BCK predicate logic is. Another novelty of Girard's paper is a new representation of proofs by what he calls *proof nets*. In [1987], Girard considers linear logic with an involutive negation, but, subsequently, intuitionistic versions of linear logic were also considered. In [Troelstra 1992], one may find a detailed survey of developments that have arisen from Girard's paper in a surprisingly short time.

6 The Lambek Calculus

In the calculus introduced by Lambek in [1958], besides Thinning and Contraction, we reject Permutation too. This gives the *associative* variant of the Lambek calculus. In this calculus, sequences of formulae in sequents are really taken seriously: there are no structural rules to reduce them to multisets or sets of formulae. The only structural rules are Identity and Cut.

In the *nonassociative* variant of the calculus, introduced by Lambek in [1961], sequents are not based on sequences of formulae, but on terms of formulae built with a binary comma, which we shall call *premise terms* (Belnap in [1982, 1990] exploits the same idea). More precisely, all formulae are premise terms, and if X and Y are premise terms, so is (X, Y). By $X[Y]$ we denote a premise term in which Y is a subterm, and $X[Z]$ is obtained from $X[Y]$ by replacing a *single* occurrence of Y by Z. In a sequent $X \vdash A$, we take X to be a premise term and not a sequence of formulae. However, these sequents will not differ essentially from single-conclusion sequents based on sequences of formulae if we assume the following structural rules:

$$\textit{Association} \qquad \frac{X[((Y_1, Y_2), Y_3)] \vdash C}{X[(Y_1, (Y_2, Y_3))] \vdash C} \qquad \frac{X[(Y_1, (Y_2, Y_3))] \vdash C}{X[((Y_1, Y_2), Y_3)] \vdash C}$$

(we might also need a premise term to stand for the empty sequence and structural rules to insert and delete this premise term). In the nonassociative Lambek calculus we reject Association. What remains of structural rules is Identity and Cut in the following form:

$$\frac{Y \vdash A \qquad X[A] \vdash C}{X[Y] \vdash C}$$

In both versions of the Lambek calculus, we have only single formulae on the right-hand side of sequents, and as connectives we have fusion $\cdot$ and two implications, $\rightarrow$ and $\leftarrow$ (Lambek's notation for the two implications is $\backslash$ and $/$). The essential principles governing these connectives are:

$$A \cdot B \vdash C \quad \textit{iff} \quad B \vdash A \rightarrow C,$$
$$A \cdot B \vdash C \quad \textit{iff} \quad \vdash C \leftarrow B.$$

Algebraically, this means that we have a left and right residual. With Permutation, the fusion connective, which stands for the comma of premise terms, becomes commutative and the two implications become equivalent.

As Thinning, Contraction and Permutation correspond to the combinators K, W and C respectively, so the Association rules correspond to the combinator B. With the first Association rule above we can prove:

(1) $A \cdot (B \cdot C) \vdash (A \cdot B) \cdot C$ and $B \rightarrow C \vdash (A \rightarrow B) \rightarrow (A \rightarrow C)$

and with the second:

(2) $(A \cdot B) \cdot C \vdash A \cdot (B \cdot C)$ and $C \leftarrow B \vdash (B \leftarrow A) \leftarrow (C \leftarrow A)$.

The implicational principles in (1) and (2) are related to the fourth implicational axiom of Hilbert's positive logic (see §1 above).

In the beginning, Lambek proposed his calculus not as a new logic but as an instrument underlying *categorial grammars* (he was also inspired by multilinear algebra and noncommutative ring theory). In this linguistic application, $A, B, C, \ldots$ stand for grammatical categories, or syntactic types, or, simply, types; $A \cdot B$ is the type of an expression obtained by concatenating an expression of type A with an expression of type B, whereas $A \rightarrow B$ and $B \leftarrow A$ are types of an expression that concatenated with an expression of type A on respectively the left-hand side and right-hand side gives an expression of type B. The sequent $A \vdash B$ means that every expression of type A is of type B.

As a linguistic tool, the Lambek calculus started attracting greater attention in the last decade. Two logicians, Buszkowski and van Benthem, have special merit for promoting investigations into the theoretical aspects of the Lambek calculus from the point of view of linguistics (see [Buszkowski et al. 1988], [Buszkowski 1989] and [van Benthem 1991] for surveys of the literature; for something closely related to the Lambek calculus in the work of the relevant logicians, see [Meyer and Routley 1972]).

Lambek's own interest in his calculus was later motivated not so much by linguistics as by category theory (in his recent papers [1988, 1989, 1989a] and in this book, he considers both aspects of the matter). The investigations of the Lambek calculus and related Gentzen-like calculuses in category theory are closely related to logic if following Lambek we understand categories as a special kind of deductive system. In a deductive system, we introduce terms that encode proofs and we are interested in relations between these terms. Categorical axioms given as identities between these terms correspond to reduction steps in a normalization procedure. These terms are also related to terms in a typed lambda calculus and the identities between terms are related to lambda conversion. This approach, which unites Gentzen-style proof theory, category theory and combinatory logic, and incorporates the Curry-Howard interpretation, is expounded for intuitionistic logic in [Lambek and Scott 1986]. For the Lambek calculus and linear logic, one may consult [Mints 1977], [Szabo 1978] and [Girard and Lafont 1987]. Presumably, this approach can be extended to other substructural logics, including relevant and BCK logic (cf. [Helman 1977], [Meredith 1978] and [Szabo 1983]). Of all these works, [Szabo 1978], which in addition to intuitionistic logic considers both the Lambek calculus and fragments of linear logic, is most open to experimenting with various structural rules and may be taken as a general work on substructural logics.

Although the Lambek calculus was not proposed as a new logic, we suppose that one may view the nonassociative Lambek calculus as a minimal multiplicative logic (multiplicative, in the sense of [Girard 1987]). From it, we pass to the stronger logics not by changing anything in the rules for fusion and implications, but only by adding structural rules. These structural rules simplify and make more manageable work with hypotheses. They enable us to consider hypotheses as given in sequences, or multisets, or sets. But if we want more complete information about hypotheses arranged at the top of proof-trees, then we should work with sequents based on premise terms that correspond to the proof-trees; and with binary proof-trees the appropriate logic should be the nonassociative Lambek calculus.

To make it more of a logic, we should extend the Lambek calculus with other connectives, as in Lambek's papers in this book. The additive connectives and the multiplicative constant t (i.e. Girard's 1) don't seem to be problematic (though they need not yield conservative extensions). However, negation may be problematic. As implication has split into two implications, so negation may split into two negations, a left one and a right one. If $\neg_L A$ and $\neg_R A$ are equivalent with respectively $A \rightarrow \bot$ and $\bot \leftarrow A$, where no postulate is assumed about $\bot$, then in the nonassociative Lambek calculus we shall have $A \vdash \neg_L \neg_R A$ and $A \vdash \neg_R \neg_L A$, which correspond to $A \vdash \neg\neg A$. To get something like an involutive negation, we should presumably consider assuming the converse sequents.

7 The Semantics of Substructural Logics

Substructural logics are proof-theoretically motivated, and their syntax seems to be better understood than their semantics. This is true even for intuitionistic logic, though model theory is incomparably more developed for intuitionistic logic than for the weaker substructural logics. In this section, I will try to say something about models for these weaker logics.

All these logics have algebraic models, but, as usual, these models are not far removed from the syntax. The core of these models serves for the multiplicative fragment and is some sort of residuated groupoid. We also have a unary operation for negation, which is involutive if negation is involutive. The lattice ordering of a residuated groupoid serves to model the additive connectives. An involutive negation operation satisfies De Morgan's laws with respect to lattice meet and join.

Somewhat farther removed from syntax, but closer to it than seems desirable, are Kripke-style models. We take as Kripke-style models something where formulae are mapped by valuations to sets of *worlds*, and relations and operations on worlds are used to formulate semantic clauses for the connectives; in other words, we are defining an algebraic model in the power set of the set of worlds. Kripke models for intuitionistic logic are so suggestive because everything is reduced to a quasi-ordering of the set of worlds and the semantic clauses for the connectives are quite natural.

In relevant logic, the standard is Kripke-style models of Routley and Meyer with a ternary relation R on worlds such that $Rxyz$ means something like $x \cdot y \leq z$, where $\cdot$ is a binary operation on worlds corresponding to fusion and $\leq$ is a partial ordering of worlds. The ternary relation R serves to model the multiplicative fragment. The semantic clause for implication is:

$$y \models A \rightarrow B \quad \textit{iff}, \quad \textit{for every } x \textit{ and } z, \textit{ if } Rxyz \textit{ and } x \models A, \textit{ then } z \models B,$$

which is related to the clause:

$$y \models A \rightarrow B \quad \textit{iff}, \quad \textit{for every } x, \textit{ if } x \models A, \textit{ then } x \cdot y \models B.$$

The involutive negation brings in an additional operation on worlds. Kripke's clauses for conjunction and disjunction produce a distributive lattice structure (this is presumably one of the reasons why relevant logicians want distribution of conjunction over disjunction). If we don't want distribution, we can introduce an additional binary operation on worlds to serve for disjunction. The whole thing starts looking *ad hoc*, and comes too close to algebra and syntax. The Kripke-style models for BCK logic of [Ono and Komori 1985] are quite close in spirit to models for relevant logic, and the phase models of [Girard 1987] don't seem to be far removed (Girard's development of Scott domains, called *coherence spaces*, in [Girard 1987] and [Girard et al. 1989] is a different matter). Standard models for the Lambek calculus are inspired by linguistics, but they are also related to the Kripke-style models of relevant, BCK and linear logic (see Ono's paper in this book and [D. 1988, 1992]).

Perhaps Kripke-style set-theoretic semantics is not what is needed for substructural logics (one may have doubts about this semantics even for intuitionism). A different sort of semantics, inspired by proof theory and category theory, as indicated in §6 above, may be more congenial to the subject matter. However, to talk about such things, we would have to abandon the historical perspective of this introduction and look into current investigations of substructural logics.

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Life in the Undistributed Middle

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Some logics have no connectives with the lattice properties. Some logics have lattice connectives satisfying distribution as well. What this paper wonders about is those logics in the middle: They have lattice connectives that do *not* satisfy the distributive laws.

By the 'lattice properties' I mean the familiar properties of conjunction and disjunction that can be stated in Gentzen fashion with no reference to structural context.

Structure-free rules

Left rules

$$\frac{A \vdash Y}{A\&B \vdash Y} \qquad \frac{B \vdash Y}{A\&B \vdash Y}$$

$$\frac{A \vdash Y \qquad B \vdash Y}{A \lor B \vdash Y}$$

Right rules

$$\frac{X \vdash A \qquad X \vdash B}{X \vdash A\&B}$$

$$\frac{X \vdash A}{X \vdash A \lor B} \qquad \frac{X \vdash B}{X \vdash A \lor B}$$

In the statement of these rules, A and B are formulas, and X and Y are perhaps complex structures. In display logic[1] these lattice rules are called 'structure-free.'

They are in contrast to rules such as the following:

Structure-dependent rules

$$\frac{A, B \vdash Y}{A\&B \vdash Y} \qquad \frac{X \vdash A \qquad Y \vdash B}{X, Y \vdash A\&B}$$

$$\frac{A \vdash X \qquad B \vdash Y}{A \lor B \vdash X, Y} \qquad \frac{X \vdash A, B}{X \vdash A \lor B}$$

1 Belnap 1982. A slightly amended version appears in Anderson and Belnap and Dunn 1992 (henceforth cited as ABD). See also Belnap 1990 for some improvements relevant to the present discussion. Another study with an entirely different flavor but with overlapping applicability is Dunn's gaggle theory.