Simulation

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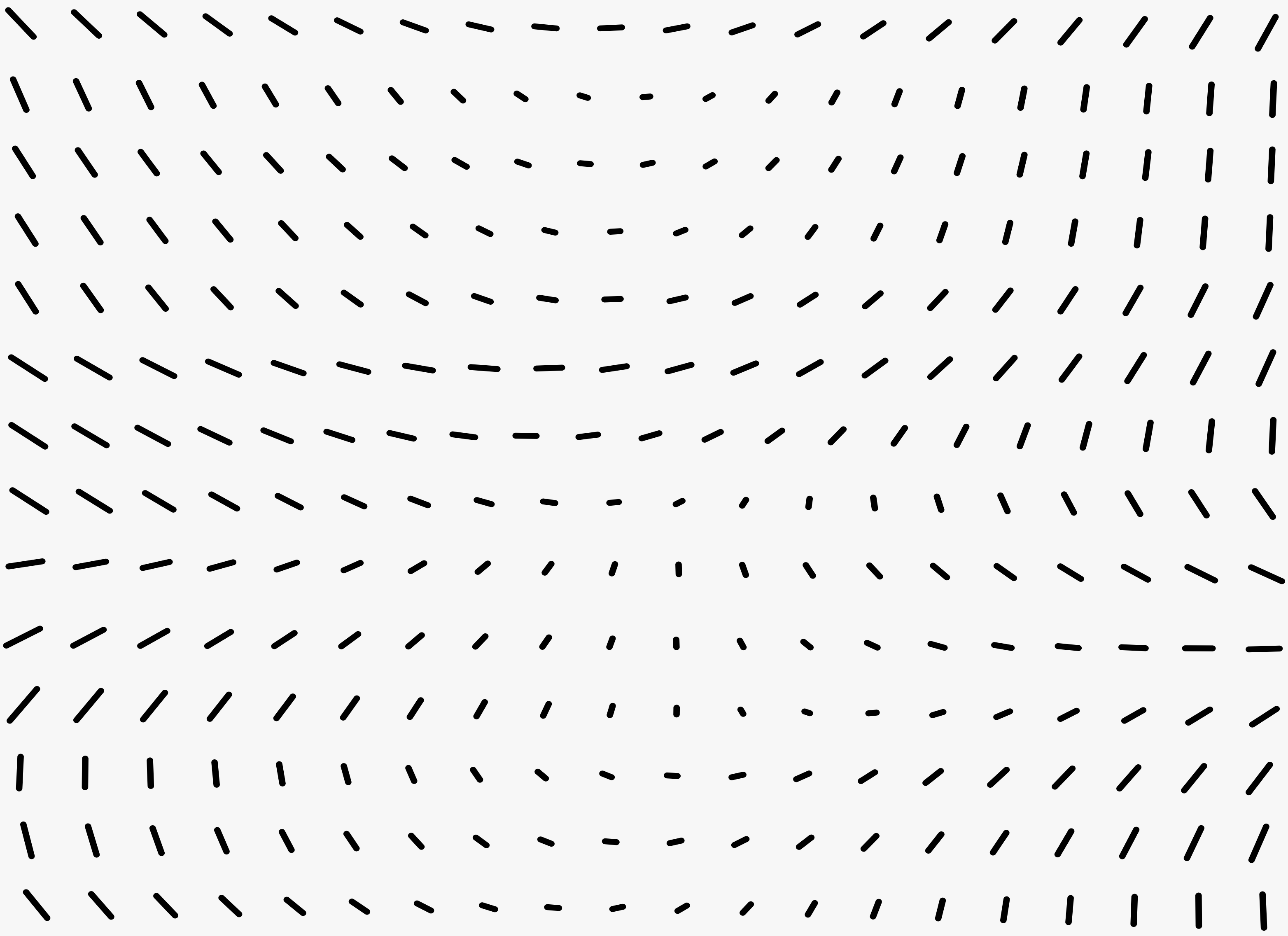


$$\theta = E[g(X)], \quad X \sim F$$

Standard-Simulation

$Y_1 = g(X_1)$

$Y_2 = g(X_2)$

$\vdots$

$Y_N = g(X_N)$

Entgegengesetzte Variable

$$Y_i = \frac{g[F^{-1}(U_i)] + \underbrace{g[F^{-1}(1-U_i)]}}{2}$$

$\hat{\theta} = \overline{Y}$

$$SE(\hat{\theta}) = \frac{SD(Y)}{\sqrt{N}}$$

$g[F_1^{-1}(U_1), \dots, F_m^{-1}(U_m)]$

$g[F_1^{-1}(1-U_1), \dots, F_m^{-1}(1-U_m)]$

Kontrollvariable

$$\theta = E[X]$$

$$Z = X + c \cdot (Y - \mu_y)$$

$$E(Z) = E(X) + c \cdot \underbrace{(\underbrace{E(Y)}_{\mu_y} - \mu_y)}_{0}$$

d.h. $E(Z) = \theta$

$$\rightarrow Var(Z) = Var(X + c \cdot (Y - \mu_y))$$

$$= Var(X) + c^2 \cdot Var(Y)$$

$$2 \cdot c \cdot cov(X, Y)$$

$$\frac{d\,Var(Z)}{dc} = 2 \cdot c \cdot Var(Y) + 2\,cov(X, Y) = 0$$

$$c^* = -\frac{\operatorname{cov}(X,Y)}{\operatorname{Var}(Y)}$$

$$\operatorname{Var}(Z) = \operatorname{Var}(X) + \frac{\operatorname{cov}(X,Y)^2}{\operatorname{Var}(Y)^{\cancel{2}}} \cdot \cancel{\operatorname{Var}(Y)}$$

$$- 2 \cdot \frac{\operatorname{cov}(X,Y)^2}{\operatorname{Var}(Y)}$$

$$= \operatorname{Var}(X) - \frac{\operatorname{cov}(X,Y)^2}{\operatorname{Var}(Y)}$$

d.h. $\operatorname{Var}(X) > \operatorname{Var}(Z)$

für $\operatorname{cov}(X,Y) \neq 0$

$$\frac{\operatorname{Var}(Z)}{\operatorname{Var}(X)} = 1 - \frac{\operatorname{cov}(X,Y)^2}{\operatorname{Var}(Y)\operatorname{Var}(X)}$$

$$= 1 - \operatorname{corr}(X,Y)^2$$

d.h. Reduktion $100 \cdot \operatorname{corr}(X,Y)^2$

Beispiel

- $$X = \frac{20}{1+U^2} \qquad U \sim (0,1)$$

- Bestimme $E(X) = \theta$
  $SE(\theta)$

- $Y = U^2$, $E(Y) = E(U^2) = \frac{1}{3}$

- $Z = X + c^* \cdot \left(Y - \frac{1}{3}\right)$

- Simulation

  1. $(X_1, Y_1), \ldots, (X_N, Y_N)$
  2. $$c^* = -\frac{\mathrm{cov}(X,Y)}{\mathrm{Var}(Y)}$$
  3. $Z_1, \ldots, Z_N$
  4. $\hat{\theta} \approx \overline{Z} \qquad SE(\hat{\theta}) = \dfrac{SD(Z)}{\sqrt{N}}$

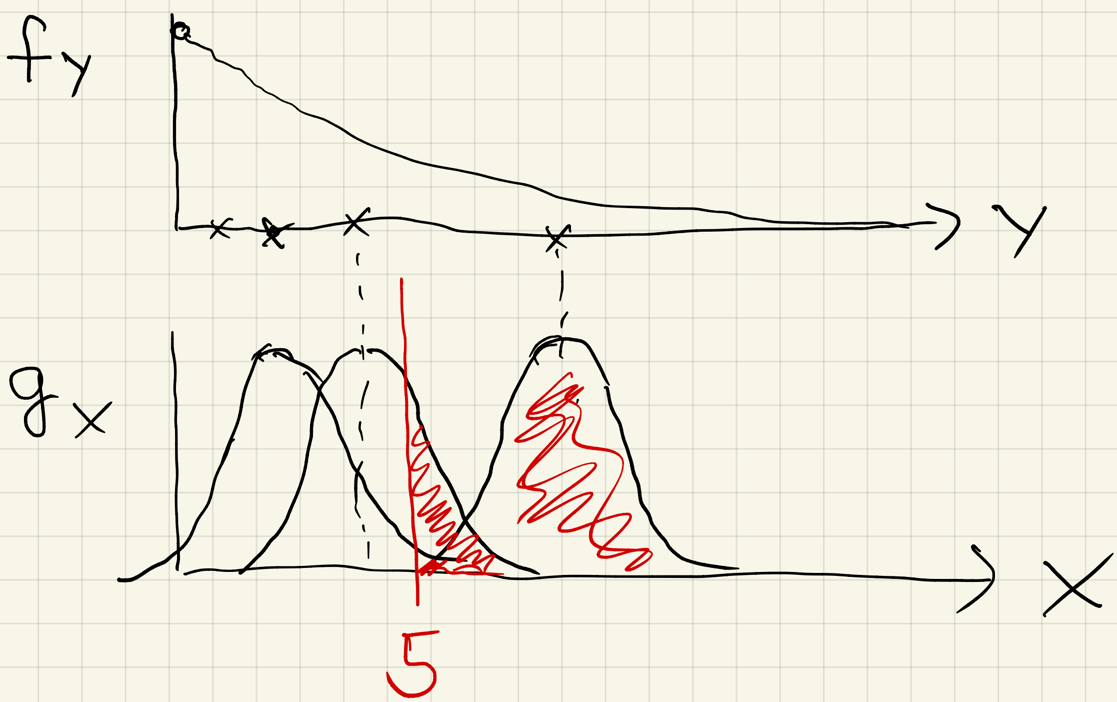
Variantreduktion durch Konditionierung

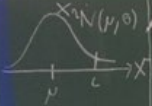
$$\theta = E(X)$$

$$E\left[E(X|Y)\right] = E(X)$$

Beispiel

$$\left.\begin{array}{l} Y \sim \exp(1) \\ X \sim N(Y, 4) \end{array}\right\} P(X > 5) = \theta$$

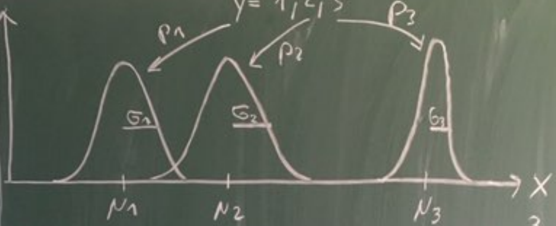




$P(X \geq c) =$

$1 - F(c) =$

$1 - \Phi\left(\frac{c-\mu}{\sigma}\right)$



$E(X \mid Y = i) = \mu_i$

$Var(X \mid Y = i) = \sigma_i$

$$E[var(X \mid Y)] = \sum_{i=1}^{3} \sigma_i \cdot p_i$$

$$Var(X) = \overset{\geq 0}{E[Var(X \mid Y)]} + \overset{\geq 0}{Var[E(X \mid Y)]}$$

$$E[E(X \mid Y)] = E(X) = \theta$$

$$Var(X) \geq Var[E(X \mid Y)]$$

$$P(X>5 \mid Y) = 1 - \Phi\left(\frac{5-Y}{4}\right)$$

$$E\left[P(X>5 \mid Y)\right] = \int_0^{\infty} \left[\ \cdots\ \right] e^{-y}\, dy$$

$$\mathrm{Var}(X) \geq \mathrm{Var}\left[E(X \mid Y)\right]$$

<u>Simulation:</u>

1. $Y_1, \ldots, Y_N$

2. $$\begin{cases} P_1 = 1 - \Phi\left(\frac{5-Y_1}{\sigma}\right) \\ \vdots \\ P_N = 1 - \Phi\left(\frac{5-Y_N}{\sigma}\right) \end{cases}$$

3. $P_1, \ldots, P_N \Rightarrow \bar{P},\ SE(\bar{p})$

Hausaufgabe   K-of-n System

$n = 30, \ k = 3$

$$S_i = \begin{cases} 1, & p_i \\ 0, & 1-p_i \end{cases}$$

$$p_i = e^{-0.3 \cdot i}, \quad i = 1, \dots, 20$$

Gesucht ist: $P\left(\sum_{i=1}^{n} S_i \geq k\right) = p$

Verwende $Y = \sum_{i=1}^{n} S_i$ als Kontroll-variable.

a) Schätze p durch Simulation mit $N = 1000$

b) Wie groß ist $SE(\hat{p})$ mit und ohne Kontrollvariable

Abgabe: 12. Juli