A brief introduction into evolutionary game theory

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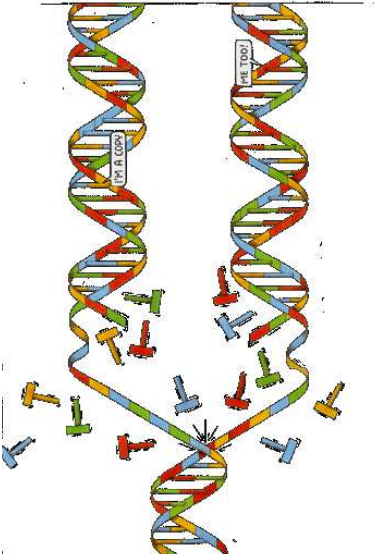
Evolutionary Game Theory

- populations of players
- individuals are (genetically) programmed for certain strategy
- individuals replicate and thereby pass on their strategy



Utility and fitness

- number of offspring is monotonically related to average utility of a player
- high utility in a competition means the outcome improves reproductive chances (and vice versa)
- number of expected offspring (Darwinian "fitness") corresponds to **expected utility** against a population of other players
- genes of individuals with high utility will spread

Extinction of non-rationalizable strategies

- strictly dominated strategies always have less-than-average reproduction rate
- their proportion thus converges towards zero
- once a strictly dominated strategies dies out (or almost dies out), it can be ignored in the utility matrix
- corresponds to *elimination of a strictly dominated strategy*
- process gets iterated in evolutionary dynamics
- long-term effect:

Theorem

If a strategy a_i is iteratively strictly dominated, then

$$\lim_{t\to\infty} p_t(a_i) = 0$$

Evolutionary stability (cont.)

- replication sometimes unfaithful (mutation)
- population is **evolutionarily stable** $\rightsquigarrow$ resistant against small amounts of mutation
- Maynard Smith (1982): static characterization of

  Evolutionarily Stable Strategies

  (ESS) in terms of utilities only
- related to Nash equilibria, but slightly different

Evolutionary stability (cont.)

Rock-Paper-Scissor

	R	P	S
R	0	-1	1
P	1	0	-1
S	-1	1	0

- one symmetric Nash equilibrium: $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$
- not evolutionarily stable though

Evolutionary stability (cont.)

Pigeon orientation game

- "players" are pigeons that go together on a journey
- A-pigeons can find their way back, B-pigeons cannot

	A	B
A	1	1
B	1	0

Evolutionary stability (cont.)

- A is a non-strict Nash equilibrium, but nevertheless evolutionarily stable
- to be evolutionarily stable, a population must be able either
  - to fight off invaders directly (strict Nash equilibrium)
  - to successfully invade the invaders (non-strict Nash equilibrium)

Evolutionary Stable Strategy

Definition

The mixed strategy α *is an* **Evolutionarily Stable Strategy** *in a symmetric two-person game iff*

- $u(\alpha, \alpha) \geq u(\alpha', \alpha)$ *for all* α, *and*
- *if* $u(\alpha, \alpha) = u(\alpha', \alpha)$ *for some* $\alpha' \neq \alpha$, *then* $u(\alpha, \alpha') > u(\alpha', \alpha')$.

Strict Nash Equilibria

$\subset$

Evolutionarily Stable Strategies

$\subset$

Nash Equilibria

Related stability notions

Definition

The mixed strategy α *is a* **Neutrally Stable Strategy** *in a symmetric two-person game iff*

- $u(\alpha, \alpha) \geq u(\alpha', \alpha)$ *for all* α*, and*
- *if* $u(\alpha, \alpha) = u(\alpha', \alpha)$ *for some* $\alpha' \neq \alpha$*, then* $u(\alpha, \alpha') \geq u(\alpha', \alpha')$.

Definition

The set of mixed strategies A *is an* **Evolutionarily Stable Set** *in a symmetric two-person game iff*

- $u(\alpha, \alpha) \geq u(\alpha', \alpha)$ *for all* α*, and*
- *if* $u(\alpha, \alpha) = u(\alpha', \alpha)$ *for some* $\alpha \notin A$*, then* $u(\alpha, \alpha') > u(\alpha', \alpha')$

Related stability notions

Some facts

- Every ESS is neutrally stable.
- Every element of an ESSet is neutrally stable.
- Every ESS forms a singleton ESSet.

The Replicator Dynamics

- implicit assumption behind notion of ESS
  - Populations are (practically) infinite.
  - Each pair of individuals is equally likely to interact.
  - The expected number of offspring of an individual (i.e., its fitness in the Darwinian sense) is monotonically related to its average utility.
- can be made explicit in a dynamic model

Replicator Dynamics (cont.)

easiest correlation between utility and fitness

$$u(i,j) = \textit{expected number of offspring of an individual of type } i \textit{ in a } j\textit{-population}$$

Suppose

- time is discrete
- in each round, each pair of players is equally likely to interact

Replicator Dynamics (cont.)

Discrete time dynamics:

$$N_i(t+1) = N_i(t) + N_i(t)(\sum_{j=1}^{n} x_j u(i,j) - d)$$

$N(t)$ ... population size at time t
$N_i(t)$ ... number of players playing strategy s_i
$x_j(t)$ ... $\frac{N_j(t)}{N(t)}$
d ... death rate

Replicator Dynamics (cont.)

generalizing to continuous time:

$$N_i(t + \Delta t) = N_i + \Delta t N_i(\sum_{j=1}^{n} x_j u(i,j) - d)$$

thus

$$\frac{\Delta N_i}{\Delta t} = N_i(\sum_{j=1}^{n} x_j u(i,j) - d)$$

Replicator Dynamics (cont.)

if $\Delta t \rightarrow 0$

$$\frac{dN_i}{dt} = N_i(\sum_{j=1}^{n} x_j u(i,j) - d)$$

Replicator Dynamics (cont.)

size of entire population may also change:

$$\begin{aligned} N(t+\Delta t) &= \sum_{i=1}^{n}(N_i + \Delta t(N_i \sum_{j=1}^{n} x_j u(i,j) - d)) \\ &= N + \Delta t(N \sum_{i=1}^{n} x_i \sum_{j=1}^{n} x_j u(i,j)) \end{aligned}$$

hence

$$\frac{dN}{dt} = N(\sum_{i=1}^{n} x_i(\sum_{j=1}^{n} x_j u(i,j) - d))$$

Replicator Dynamics (cont.)

let

$$\sum_{j=1}^{n} x_j u(i,j) = \tilde{u}_i$$
$$\sum_{i=1}^{n} x_i \tilde{u}_i = \tilde{u}$$

then we have

$$\frac{dN_i}{dt} = N_i(\tilde{u}_i - d)$$
$$\frac{dN}{dt} = N(\tilde{u} - d)$$

Replicator dynamics (cont.)

remember some calculus?

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

Replicator dynamics (cont.)

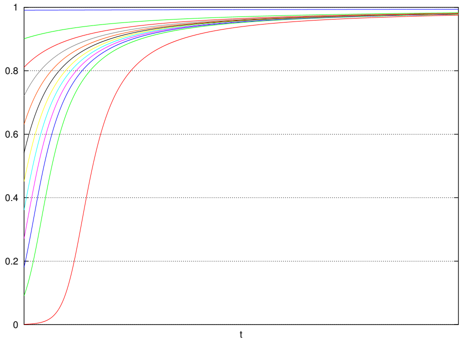
remember some calculus?

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

$$\begin{aligned} \frac{dx_i}{dt} &= \frac{(NN_i(\tilde{u}_i - d) - (N_i N(\tilde{u} - d)))}{N^2} \\ &= x_i(\tilde{u}_i - \tilde{u}) \end{aligned}$$

Pigeon orientation

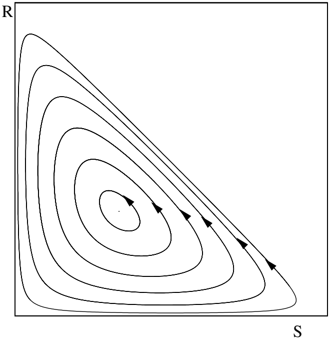
- each ESS is an **asymptotically stable state** (in finite games, that is...)
- inverse does not always hold (but we will only consider games where it does)
- a.k.a. **point attractors**
- sample dynamics:



x-axis: time
y-axis: proportion of A-players

Rock-Paper-Scissor again

- three-strategy game: two independent variables
  - number of R-players
  - number of P-players
- number of S-players follows because everything sums up to 1
- supressing time dimension gives **orbits**



Asymmetric games

- symmetric games:
  - same strategy set for both players
  - $u_A(i, j) = u_B(j, i)$ for all strategies s_i, s_j
  - evolutionary interpretation: symmetric interaction *within one population*
- asymmetric games:
  - players have different strategy sets or utility matrices
  - evolutionary interpretation
    - different roles within one population (like incumbent vs. intruder, speaker vs. hearer, ...), or
    - interaction between disjoint populations
- evolutionary behavior differs significantly!

Asymmetric games (cont.)

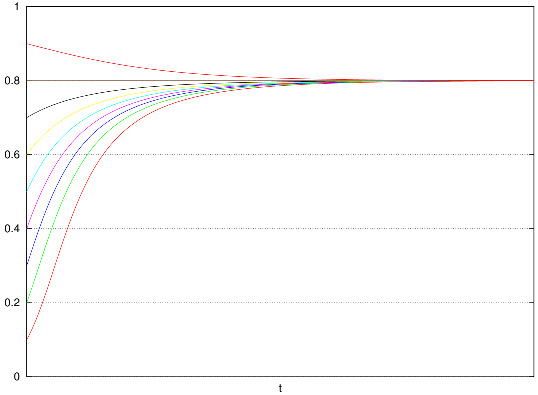
Hawks and Doves

	H	D
H	1,1	7,2
D	2,7	3,3

- can be interpreted symmetrically or asymmetrically
- symmetric interpretation:
  - hawks prefer to interact with doves and vice versa
  - ESS: 80% hawks / 20% doves
  - both strategies have average utility of 2.2
  - dynamics:

Symmetric Hawk-and-doves

- if hawks exceed 80%, doves thrive, and vice versa
- 80:20 ratio is only attractor state

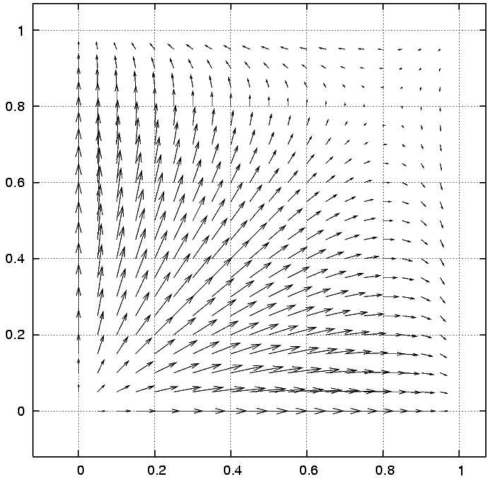


Asymmetric Hawks-and-doves

- suppose two-population setting:
  - both A and B come in hawkish and dovish variant
  - everybody only interacts with individuals from opposite "species"
  - excess of A-hawks helps B-doves and vice versa
  - population push each other into opposite directions

Hawks and doves

- 80:20 ratio in both populations is stationary
- not an attractor, but repellor



Asymmetric stability

- crucial difference to symmetric games:
  mutants do not play against themselves
- makes second clause of the symmetric ESS superfluous

Theorem (Selten 1980)

In asymmetric games, a configuration is an ESS iff it is a strict Nash equilibrium.

Asymmetric replicator dynamic

$$\frac{dx_i}{dt} = x_i(\sum_{j=1}^{n} y_j u_A(i,j) - \sum_{k=1}^{n} x_k \sum_{j=1}^{n} y_j u_A(k,j))$$

$$\frac{dy_i}{dt} = y_i(\sum_{j=1}^{m} x_j u_B(i,j) - \sum_{k=1}^{n} y_k \sum_{j=1}^{m} x_j u_B(k,j))$$

x_i ... proportion of s_i^A within the A-population
y_i ... proportion of s_i^B within the B-population

Symmetrizing asymmetric games

- asymmetric games can be "symmetrized"
- correspondig symmetric game shares Nash equilibria and ESSs
- new strategy set:

$$S^{AB} = S^A \times S^B$$

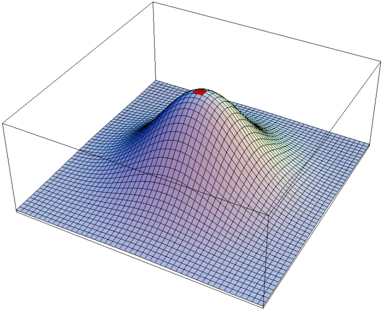
- new utility function

$$u^{AB}(\langle i, j\rangle, \langle k, l\rangle) = u^A(i, l) + u^B(j, k)$$

Stability in symmetrized games

strict Nash equilibria

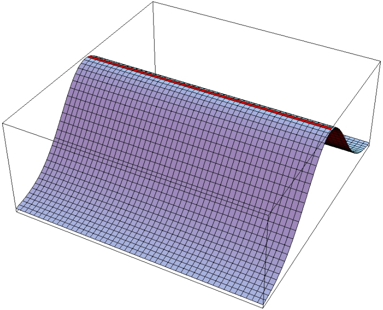
In symmetrized games, the asymptotically stable states are exactly the strict Nash equilibria. (Selten 1980)



Stability in symmetrized games

neutrally stable states

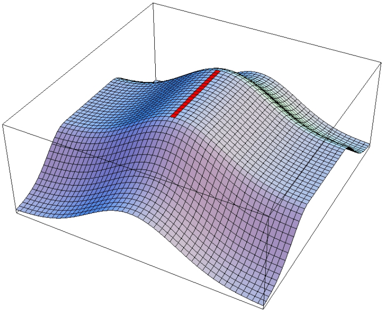
In symmetrized games, a strategy is Lyapunov stable iff it is a neutrally stable state. (Cressman 2003)



Stability in symmetrized games

ESSets

In symmetrized games, a set of strategies is an asymptotically stable set of rest points iff it is an ESSet.



Exercises

1. Find the symmetric ESSs of the following games (provided they exist):
    - Prisoner's dilemma
    - Stag hunt
2. Find the asymmetric ESSs of the following games (again, provided they exist):
    - Bach or Stravinsky
    - Matching pennies
3. Symmetrize the asymmetric version of Hawks and Doves and find the symmetric ESSs of the result. Which configuration in the original game do they correspond to?