Mathematics for linguists

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Sets

Georg Cantor (1845-1918)

"A set is a collection into whole of definite, distinct objects of our intuition or our thought. The objects are called the elements of the set."

- Every well-defined object can be member/element of a set
- Sets can be members of other sets.
- The question of membership must be answerable in principle.
- Sets can be finite or infinite.

Sets

- special sets
  - singleton sets (contain exactly one element): $\{a\}$
  - the empty set (contains no element): $\emptyset$ (also written as 0 or $\{\}$)
- notational conventions:
  - $A, B, C, \ldots$: variables over sets
  - $a, b, c, \ldots, x, y, z$: variable over elements of sets
  - $a \in A$: a is an element of A
  - $a \notin A$: a is not an element of A
  - *important:* since sets can be elements of other sets, we sometimes find expressions like $A \in B$

Ways to describe sets

- four ways to describe sets
  - list notation
  - separation notation
  - recursive definition
  - set theoretic operations

Ways to describe sets: List notation

- only applicable to finite sets
- names of the elements are listed between curly brackets
- example:

$$A = \{\textit{the Volga, Nicolas Sarkozy, 16}\}$$

- can alseo be written as

$$A = \{\textit{Europe's longest river, the French president, the number of federal states in Germay}\}$$

- order is irrelevant:

$$A = \{\textit{16, the Volga, Nicolas Sarkozy}\}$$

Ways to describe sets: List notation

- it is also inessential how often an object is named in list notation

  $A = \{$*the president of France, Nicolas Sarkozy, the winner of the last presidential election in France*, 4^2, 16, *the Volga*, $\sqrt{256}\}$

Ways to describe sets: separation notation

- Set of all objects of a domain that share a certain property
- domain must also be a set
- domain has to be well-defined, before it can be used to define other sets
- notation:

  $$\{ \text{ variable} \in \text{domain} \mid \text{sentence that contains the variable } \}$$

  or

  $$\{ \text{ variable} \in \text{domain} : \text{sentence that contains the variable } \}$$

Ways to describe sets: separation notation

- examples:
  - $\{x \in \mathbb{N} | x \text{ is even}\}$
  - $\{x \in \mathbb{N} | x - 10 \geq 0\}$
  - $\{x \in \mathbb{R} | x^2 = 2\}$
- domain is frequently omitted if it is clear from the context

Russell's paradox

Why is it so important to always specify a domain when definint a set? The English philosopher Bertrand Russell showed in 2001 that otherwise (via so-called "unconstrained comprehension") it is possible to derive contradictions. For instance, consider:

$$R = \{x | x \notin x\}$$

Does the following hold:

$$R \in R?$$

Suppose $R \in R$. Then R must have the defining property, i.e. $R \notin R$. On the other hand, if $R \notin R$, then R has the defining property, and thus $R \in R$. In either case, we end up with a contradiction.

It doesn't help to prohibit that a set contains itself. Then R would be the set of all sets, which would have to contain itself after all, so we would end up with a

Russell's paradox

In modern set theory, it is usually assumed that (contra Cantor) not every collection of well-defined object automatically constitutes a set. Whether or not a collection of objects is a set is something which has to be proved. In particular, the collection of all sets is itself not a set.

The four ways to define sets that are discussed here only produce collections that are, in fact, provably sets.

Ways to describe sets: recursive definition

- consists of three components:
  1. a finite list of objects that definitels belong to the set to be defined
  2. rules that allow to generate new elements from existing elements
  3. statement, that all elements of the set in question can be generated via finitely many application of the rule from (2) to the objects from (1)

Spezifikation von Mengen: rekursive Definition

- example:
  1. $4 \in E$
  2. If $x \in E$, then $x + 2 \in E$.
  3. Nothing else is in E.

  alternative Definition via separation:
  $\{x \in \mathbb{N} | x \text{ is even and } x \geq 4\}$
- another example:
  - Genghis Khan $\in D$
  - If $x \in D$ and y is a son of x, then $y \in D$.
  - Nothing else is in D.

  D is the set which consists of Genghis Khan and all its male descendents.

Set theoretic operations: set union

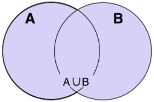
- $A \cup B$: *set unipon* (or just "union") of A and B
- set of all objects that are element of A or of B (or both)
- example: Let $K = \{a, b\}$, $L = \{c, d\}$ and $M = \{b, d\}$

$$
\begin{array}{lclcl}
K \cup L & = & \{a, b, c, d\} & & \\
K \cup M & = & \{a, b, d\} & & \\
L \cup M & = & \{b, c, d\} & & \\
(K \cup L) \cup M & = & K \cup (L \cup M) & = & \{a, b, c, d\} \\
K \cup \emptyset & = & \{a, b\} & = & K \\
L \cup \emptyset & = & \{c, d\} & = & L
\end{array}
$$

- If A is a set of sets, we write $\bigcup A$ for the union of all elements of A. Instad of $B \cup C$ we could also write $\bigcup\{B, C\}$.

Set theoretic operations: set union

graphical representatio in a Venn diagram



Set theoretic operations: set intersection

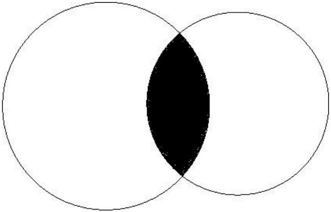
- $A \cap B$: intersection of A and B
- set of all objects, that are both member of A and of B
- example: Let $K = \{a, b\}$, $L = \{c, d\}$ and $M = \{b, d\}$

$$
\begin{array}{lclcl}
K \cap L & = & \emptyset & & \\
L \cap M & = & \{d\} & & \\
K \cup K & = & \{a, b\} & = & K \\
K \cap \emptyset & = & \emptyset & & \\
(K \cap L) \cap M & = & K \cap (L \cap M) & = & \emptyset \\
K \cap (L \cup M) & = & \{b\} & &
\end{array}
$$

- Intersection can be generalized to all sets of sets as well. $\bigcap A$ is the set of all objects, that are a member of all members of A.

Set theoretic operations: set intersection

representation in Venn diagram



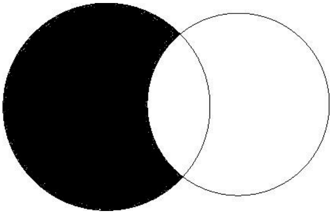
Set theoretic operations: relative complement

- $A - B$ (also written as $A \setminus B$: relative complement of B in A
- set of all objects, that are an element of A, but not of B
- examples: Let $K = \{a, b\}$, $L = \{c, d\}$ and $M = \{b, d\}$

$$\begin{array}{lllll} K - M & = & \{a\} & & \\ L - K & = & \{c, d\} & = & L \\ M - L & = & \{b\} & & \\ K - \emptyset & = & \{a, b\} & = & K \\ \emptyset - K & = & \emptyset & & \end{array}$$

Set theoretic operations: relative complement

representation in Venn diagram

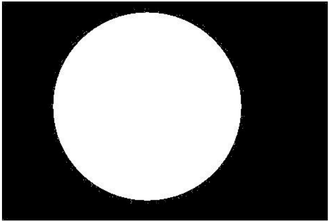


Set theoretic operations: absolute complement

- A' (also written as $\overline{A}$ or $-A$: absolute complement of A
- set of all objects that are not element of A
- only well-defined against the background of a (usually implicit) universe U
- more precise notation: $U - A$

Set theoretic operations: relative complement

representation in Venn diagram



Identity of sets

- the same set can be defined in different ways
- for instance:
  1. $A = \{1, 2, 3, 4, 5\}$
  2. $A = \{x \in \mathbb{N} | x > 0$ und $x < 6\}$
  3. $1 \in A;$ if $x \in A$ and $x < 5,$ then $x + 1 \in A$; nothing else is in A

  When do two descriptions define the same set?

Identity of sets

Two sets are identical if and only if they have the same elements.

In other words, $A =$B iff every element of A is also an element of B, and every element of B is also an element of A.

Subsets

- $A \subseteq B$: A is a subset B
- Every element of A is also an element of B.
- B may contain more elements than those from A, but this need not be the case
- $A \not\subseteq B$: A is not a subset of B.
- $A \subset B$: A is a *proper* subset of B
- A is a subset of B, and B contains at least one element that is not in A
- Equivalently:

$$A \subset B \text{ iff } A \subseteq B \text{ and } B \not\subseteq A$$

Subsets

- $A \supseteq B$: A is a superset of B
- $A \supseteq B$ iff $B \subseteq A$

Subsets

Beispiele:

1. $\{a, b, c\} \subseteq \{s, b, a, e, g, i, c\}$
2. $\{a, b, j\} \not\subseteq \{s, b, a, e, g, i, c\}$
3. $\{a, b, c\} \subset \{s, b, a, e, g, i, c\}$
4. $\emptyset \subset \{a\}$
5. $\{a, \{a\}\} \subseteq \{a, b, \{a\}\}$
6. $\{a\} \not\subseteq \{\{a\}\}$
7. $\{\{a\}\} \not\subseteq \{a\}$
8. $\emptyset \subseteq A$ for arbitrary sets A
9. but: $\{\emptyset\} \not\subseteq \{a\}$

Note:

The element-of relation and the subset-relation have to be clearly distinguished!

for instance:

$$\begin{array}{ccc} a & \in & \{a\} \\ a & \not\subseteq & \{a\} \end{array}$$

or

$$\begin{array}{ccc} \{a\} & \subseteq & \{a,b,c\} \\ \{a\} & \notin & \{a,b,c\} \end{array}$$

Subsets

Note:

Subset relation is transitive:
If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Set theoretic operations: power set

- $\wp(A)$ (sometimes written $POW(A)$, $Pot(A)$ or 2^A): power set of A
- set of all subsets A
- exmples:
  1. $\wp(\{a, b, c\}) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$
  2. $\wp(\emptyset) = \{\emptyset\}$
  3. $\wp(\wp(\emptyset)) = \{\emptyset, \{\emptyset\}\}$
  4. $\wp(\{a\}) = \{\emptyset, \{a\}\}$
- If A is finite and has n elements, then $\wp(A)$ always has 2^n elements.
- For all sets A: $\emptyset \in \wp(A)$ and $A \in \wp(A)$.

Cardinality of sets

- $|A|$ (sometimes also written als $\#(A)$): cardinality of A
- for empty sets: $|A|$ is the number of elements of A
- examples:
  1. $|\emptyset| = 0$
  2. $|\{a\}| = 1$
  3. $|\{\emptyset\}| = 1$
  4. $|\{a, \{b, c, d\}\}| = 2$
- cardinality is also defined for infinite sets
- $|A| = |B|$ iff there is a one-one mapping between A and B. (The notion of a one-one mapping will be introduced later in this course.)
- Not all infinite sets have the same cardinality

Set theoretic laws

1. idempotence laws:
   1. $A \cup A = A$
   2. $A \cap A = A$
2. commutativity laws:
   1. $A \cup B = B \cup A$
   2. $A \cap B = B \cap A$
3. associativity laws:
   1. $(A \cup B) \cup C = A \cup (B \cup C)$
   2. $(A \cap B) \cap C = A \cap (B \cap C)$
4. distributivity laws:
   1. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   2. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Set theoretic laws

1. identity laws:
   1. $A \cup \emptyset = A$
   2. $A \cup U = U$
   3. $A \cap \emptyset = \emptyset$
   4. $A \cap U = A$
2. complement laws:
   1. $A \cup A' = U$
   2. $(A')' = A$
   3. $A \cap A' = \emptyset$
   4. $A - B = A \cap B'$
3. De Morgan's laws:
   1. $(A \cup B)' = A' \cap B'$
   2. $(A \cap B)' = A' \cup B'$
4. consistency principle:
   1. $A \subseteq B$ gdw. $A \cup B = B$
   2. $A \subseteq B$ gdw. $A \cap B = A$