Mathematics for linguists

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Sentences and statements

(1) You again!

(2) Has the lecture already started?

(3) Could you please tell me how I can get to the bus stop of the line 4?

(4) One year ago, there were 891 students of philosophy at the University of Bielefeld.

- These are all grammatical sentences.
- logically interesting sentences must be able to have a **truth value**
- Question: Which example sentence can, in principle, have a truth value?

Sentences and statements

- Truth value of (4) depends on time of utterance
- similar examples

(5) Charlemagne was appointed emperor *here* in 800 A.D.

(6) *Today* is Tuesday.

(7) John is coming over *there*.

(8) The window is *to the left* of the door.

Sentences and statements

- Truth value of these sentences depends on **utterance situation**
- in other words: in different utterance situations, these sentences express different **statements**
- responsible for this effect: **deictic** (also called **indexical** expressions *here, there, to the left, to the right, now, tomorrow, last year, I, you, ...*)

Sentences and statements

- Statements are sentences that are, in principle, either true or false.
- Statement logic and predicate logic only deal with statements the truth of which do not depend on the situation in which they are uttered.

Sentences and statements

Which of the following sentences express statements in the sense of statement logic?

(9) The Zugspitze is Germany's highest mountain.

(10) I have shown you his letter, here and today.

(11) Please give me the salt!

(12) Did you sleep well?

(13) How utterly beautiful!

Statement logic: connectives

- linguistic means to construct new statements out of smaller statements
- examples:

by no means; and; but; Peter knows, that; or; if ... then; if and only iff; perhaps

Negation

- If a statement is true, its negation is false (and vice versa).
- Statement

(14) Peter is in Berlin.

- Possible expressions for negation:

(15) a. Peter is not in Berlin
b. It is not the case that Peter is in Berlin.
c. It is not true that Peter is in Berlin.

- schematically:
  - statement: φ
  - negation: $\neg\varphi$

Negation

- truth values: “true” and “false”
- schematically: 1 (for “true”) and 0 (for “false”)
- **truth table** for negation

φ	$\neg\varphi$
1	0
0	1

Conjunction

- conjunction: combines two statements
- conjunction is true if and only if both conjuncts are true
- e.g.: statements:

(16) a. Wolfgang sleeps.
     b. Wolfgang snores.

- conjunktion:

(17) a. Wolfgang sleeps and Wolfgang snores.
     b. Wolfgang sleeps and snores.
     c. Wolfgang sleeps, and he also snores.
     d. Wolfgang both sleeps and snores.
     e. ...

Conjunction

- schematically
  - statements: φ, ψ
  - conjunction: $\varphi \wedge \psi$
- truth table:

φ	ψ	$\varphi \wedge \psi$
1	1	1
1	0	0
0	1	0
0	0	0

Disjunction

- disjunction: combines two statements
- is true if and only if at least one of the two components is true
- e.g.:

(18) a. It rains.
b. It is dark.

- disjunction:

(19) It rains or it is dark.

Disjunction

- xchematically:
  - statements: φ, ψ
  - disjunction: $\varphi \vee \psi$
- truth table:

φ	ψ	$\varphi \vee \psi$
1	1	1
1	0	1
0	1	1
0	0	0

Inclusive and exclusive *or*

- disjunction corresponds to **inclusive "or"**
- intended meaning:

(20) a. It rains and/or it is dark.
b. It rains or it is dark or both.

- there is also an **exclusive** sense of "or"
- "or" can mean either in German
- example:

(21) a. Either we go to the movies or to the zoo.
b. We either go to the movies or to the zoo, but not both.

XOR

- exclusive or is modeled by different logical operator: **XOR**
- schematically: $\varphi \infty \psi$
- truth table:

φ	ψ	$\varphi \infty \psi$
1	1	0
1	0	1
0	1	1
0	0	0

- Unless otherwise indicated, we will henceforth use “or” in the sense of disjunction.

Implication

- statements: φ, ψ
- implication: $\varphi \rightarrow \psi$
- to be read as: „φ **implies** ψ“
- truth table:

φ	ψ	$\varphi \rightarrow \psi$
1	1	1
1	0	0
0	1	1
0	0	1

Implication

- related to conditional construction

If ..., then ...

- For all four truth value combination there are corresponding English examples with conditional construction:
  1. (true – true): *If 2+2=4, then 2+3=5.* (true)
  2. (true – false): *If 2+2=4, then 2+3=4.* (false)
  3. (false – true): *If 1=2 and 2=1, then 3=3.* (true)
  4. (false – false): *If 1=2, then 2=3.* (true)

Implication

Bertrand Russell, in a lecture on logic, mentioned that in the sense of material implication, a false proposition implies any proposition. A student raised his hand and said „In that case, given that 1 = 0, prove that you are the Pope“. Russell immediately replied, „Add 1 to both sides of the equation: then we have 2 = 1. The set containing just me and the Pope has 2 members. But 2 = 1, so it has only 1 member; therefore, I am the Pope.“

Implication

- If there is no intrinsic connection between the content of φ and ψ, an implication may be true while the corresponding conditional statement is at least questionable.

(22) a. *If 1 = 0, then Bertrand Russell is the pope.* (unclear whether true or false)
b. *1 = 0 implies that Bertrand Russell is the pope.* (true)

(23) a. *If the moon is made from green cheese, then it is made from chocoloate.* (probably false)
b. *That the moon is made from green cheese implies that it is made from chocolade.* (true)

Equivalence

- statements: φ, ψ
- combined statement: $\varphi \leftrightarrow \psi$
- to be read as „φ **is equivalent to** ψ“

φ	ψ	$\varphi \leftrightarrow \psi$
1	1	1
1	0	0
0	1	0
0	0	1

Equivalence

- relatedc to the English constructions
  - φ *if and only if* ψ
  - φ *just in case that* ψ
  - φ *is a necessary and sufficient condition for* ψ
- similar provisos apply as to the identification of the implication with the conditional construction[a]

[a] Please note that Partee et al. use the terms “conditional” and “biconditional” for implication and equivalence respectively. For the reasons mentioned above, I prefer the more neutral terms.

Syntax of statement logic

A language of statement logic consists of a (usually infinite) set of atomic statements. These are statements that do not consist of statements themseleves.

- *Peter sleeps* is an atomic statement.
- *Peter smiles when he sleeps* is not an atomic statement.

Since the internal structure of atomic statements is irrelevant for statement logic, we mostly use symbols like

$$p, q, r, p_1, q_5, r', r'', ..$$

as atomic statements ("statement variables").

Syntax of statement logic

Definition 1 *Let $\mathcal{A}$ be a set of atomic statements.*

1. *Every statement in $\mathcal{A}$ is a formula $L(\mathcal{A})$.*
2. *If ψ is a formula in $L(\mathcal{A})$, then $\neg\psi$ is also a formula in $L(\mathcal{A})$.*
3. *If φ and ψ are formulas in $L(\mathcal{A})$, then $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \rightarrow \psi)$ and $(\varphi \leftrightarrow \psi)$ are also formulas in $L(\mathcal{A})$.*
4. *There are no other formulas in $L(\mathcal{A})$.*

Syntax of statement logic

Which of the following expressions are formulas of a language of statement logic?

$\neg(\neg p \vee q)$	$p \vee (q)$
$\neg(q)$	$(p_2 \rightarrow (p_2 \rightarrow (p_2 \rightarrow p_2)))$
$(p \rightarrow ((p \rightarrow q)))$	$((p \rightarrow p) \rightarrow (q \rightarrow q))$
$((p_{28} \rightarrow p_3) \rightarrow p_4)$	$(p \rightarrow (p \rightarrow q) \rightarrow q)$
$(p \vee (q \vee r))$	$(p \vee q \vee r)$
$(\neg p \vee \neg\neg p)$	$(p \vee p)$

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$((p_{28} \rightarrow p_3) \rightarrow p_4)$	~~$(p \rightarrow (p \rightarrow q) \rightarrow q)$~~
$(p \vee (q \vee r))$	~~$(p \vee q \vee r)$~~
$(\neg p \vee \neg\neg p)$	$(p \vee p)$

Bracketing conventions

- redundant brackets can be omitted
- Conventions:
  - outermost brackets are omitted
  - $\neg$ associates strongest, followed by $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$ (in this order)
  - operators are **right associative:**

$$p \rightarrow q \rightarrow r = (p \rightarrow (q \rightarrow r))$$

Semantics of statement logic

- **valuation function V:** Function that assigns each formula of a language of propositional logic a truth value
- admissible valuation functions must agree with the interpretation of the logical connectives:

Semantics of statement logic

Definition 3 *A function* V *from the formulas of a language of statement logic* $L(\mathcal{A})$ *into the set of truth values* $\{0, 1\}$ *is a* ***valuation function*** *iff it holds for all formula* φ *and* ψ*:*

1. $V(\neg\varphi) = 1 - V(\varphi)$
2. $V(\varphi \wedge \psi) = V(\varphi) \times V(\psi)$
3. $V(\varphi \vee \psi) = V(\varphi) + V(\psi) - V(\varphi) \times V(\psi)$
4. $V(\varphi \rightarrow \psi) = 1 - V(\varphi) \times (1 - V(\psi))$
5. $V(\varphi \leftrightarrow \psi) = 1 - (V(\varphi) - V(\psi))^2$

Semantics of statement logic

- every operator corresponds to a function over truth values
- arithmetic defintion is equivalent to the truth table given above
- truth value of a complex formula φ under V is uniquele determined by the truth values of the atomic statements that occur in φ under V

Semantics of statement logic

- to compute the **truth conditions** of a complex formula φ,
  - it is not necessary to consider all conceivable valuation functions, but
  - only all possible combinations of truth values of the atomic statements that occur in φ, i.e.
  - 2^n different combinations of truth values, for n atomic statements.

Semantics of statement logic

	p	q	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$\neg(\neg p \wedge \neg q)$
V_1	1	1				

Semantics of statement logic

	p	q	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$\neg(\neg p \wedge \neg q)$
V_1	1	1	0			

Semantics of statement logic

	p	q	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$\neg(\neg p \wedge \neg q)$
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Semantics of statement logic

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Semantics of statement logic

	p	q	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$\neg(\neg p \wedge \neg q)$
V_1	1	1	0	0	0	1
V_2	1	0	0	1	0	1
V_3	0	1	1	0	0	1
V_4	0	0	1	1	1	0