Mathematics for linguists

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University of Tübingen

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Gerhard Jäger

Logical inference

Definition 6 (Inference) *A formula φ **follows logically** from a set of formulas M — formally written as*

$$M \Rightarrow \varphi$$

if and only if it holds for all valuation functions V: If for all $\psi \in M$:

$$V(\psi) = 1$$

then

$$V(\varphi) = 1$$

Logical inference

- If $M \Rightarrow \varphi$, this is also called a **valid argument**.
- M is called the **set of premises** and φ the **conclusions**
- tautologies logically follow from the empty set
- examples for valid arguments

$$
\begin{aligned}
p, q &\Rightarrow p \\
p, q &\Rightarrow p \wedge q \\
p \wedge q &\Rightarrow q \wedge p \\
p, q &\Rightarrow q \vee r \\
p &\Rightarrow q \rightarrow p \\
p, p \rightarrow q &\Rightarrow q \\
p \rightarrow q, q \rightarrow r &\Rightarrow p \rightarrow r
\end{aligned}
$$

Logical inference

- for finite M, validity of an argument can be decided with the help of truth tables
- In each line where each premise has the truth value “1”, the conclusion must have the truth value “1” as well.
- **example:** „Modus Ponens“

$$p, p \rightarrow q \Rightarrow q$$

Logical inference

	p	q	$p \rightarrow q$
V_1	1	1	1
V_2	1	0	0
V_3	0	1	1
V_4	0	0	1

Logical inference

	p	q	$p \rightarrow q$
V_1	1	1	1
V_2	1	0	0
V_3	0	1	1
V_4	0	0	1

Only in the first line all premises are true, and there the conclusion is also true.

The deduction theorem

Theorem 5 *For arbitrary formulas* $\varphi_1, \cdots, \varphi_n, \psi$*,*

$$M, \varphi_1, \cdots, \varphi_n \Rightarrow \psi$$

if and only if

$$M \Rightarrow \varphi_1 \rightarrow \cdots \rightarrow \varphi_n \rightarrow \psi$$

Proof: We prove the theorem via complete induction over n.

- *Induction base* $n = 0$*:* The theorem obviously holds.

The deduction theorem

- *Induction step, forward direction:* Suppose the theorem holds for n. We have to show that in this case, it also holds for $n+1$. Let us furthermore assume that for all $\xi \in M, V(\xi) = 1$ for some arbitrary valuation function V. Now there are two alternatives:
  1. $V(\varphi_1) = 0$. It follows from the semantics of implication that in this case, $V(\varphi_1 \rightarrow \cdots \rightarrow \varphi_n \rightarrow \varphi_{n+1} \rightarrow \psi) = 1$.
  2. $V(\varphi_1) = 1$. It follows from the induction assumption that $M, \varphi_1 \Rightarrow \varphi_2 \rightarrow \cdots \rightarrow \varphi_n \rightarrow \varphi_{n+1} \rightarrow \psi$. Therefore $V(\varphi_2 \rightarrow \cdots \rightarrow \varphi_n \rightarrow \varphi_{n+1} \rightarrow \psi) = 1$. According to the semantics of implication, it also holds that $V(\varphi_1 \rightarrow \cdots \rightarrow \varphi_n \rightarrow \varphi_{n+1} \rightarrow \psi) = 1$.

  So from the assumption that V verifies the premises it follows in both cases that it alsow verifies the conclusion.

The deduction theorem

- *Induction step, backward direction:* Suppose the theorem holds for n. Let us furthermore assume that $M \Rightarrow \varphi_1 \rightarrow \cdots \rightarrow \varphi_{n+1} \rightarrow \psi$. Finally we also assume thet for all $\xi \in M : V(\xi) = 1$, and $V(\varphi_i) = 1$ for $1 \leq i \leq n + 1$. According to the induction assumption, it holds that: $M, \varphi_1, \ldots, \varphi_n \Rightarrow \varphi_{n+1} \rightarrow \psi$. Hence $V(\varphi_{n+1} \rightarrow \psi) = 1$. Due to the semantics of the implication, we also have $V(\psi) = 1$.

$\dashv$

The deduction theorem

- The deduction theorem is the basis for the method of the **conditional proof**.
- To prove that *If* A *then* B is logically true (or follows from a set of background premises), you assume
  - A as and additional premise, and
  - prove B with the help of this premise.

The truth tree method

- Proof via truth tables if often tedious and reduntant
- alternative: **indirect proof**
- You start with the assumption that an argument is invalid, and you try to derive a contradiction.
- The argument is not valid if there is at least one valuation function V that makes all premises true and the conclusion false.

The truth tree method

- Example:

$$\Rightarrow p \rightarrow q \rightarrow r \vee p$$

  - no premises; if conclusion is false, $\neg(p \rightarrow q \rightarrow r \vee p)$ must be true
  - hence p must be true and $q \rightarrow r \vee p$ false
  - hence $\neg(q \rightarrow r \vee p)$ must be true
  - hence q must be true and $r \vee p$ must be false
  - hence $\neg(r \vee p)$ must be true
  - hence both r and p must be false
  - **contradiction**
- The assumption that the formula is not a tautology led to a contradiction
- formula is thus a tautology

The truth tree method

- schematic representation in a tree

1.	$\neg(p \rightarrow q \rightarrow r \vee p)$	(A)
2.	p	(1)
3.	$\neg(q \rightarrow r \vee p)$	(1)
4.	q	(3)
5.	$\neg(r \vee p)$	(3)
6.	$\neg r$	(5)
7.	$\neg p$	(5)
8.	x	$(1, 7)$

- here: degenerate tree that is non-branching
- in the general case, truth trees may be branching

The truth tree method

- each line consists of
  1. line number
  2. formula that is assumed to be true
  3. number of the line from which the current line is derived (the first line is called “assumption” (A))
- If a branch
  - contains the formula φ which is dominated by $\neg\varphi$, or
  - contains the formula $\neg\varphi$ which is dominated by φ,

  then this branch is marked as contradictory with an “x”.
- A truth tree is **closed** if all branches are contradictory, i.e. all leaves are marked with “x”.

Further example

1.	$\neg((p \to q) \to (q \to r) \to p \to r)$	(A)
2.	$p \to q$	(1)
3.	$\neg((q \to r) \to p \to r)$	(1)
4.	$q \to r$	(3)
5.	$\neg(p \to r)$	(3)
6.	p	(5)
7.	$\neg r$	(5)

8. $\neg p$ (2)
   × $(6, 8)$

9. q (2)

10. $\neg q$ (4)
    × $(9, 10)$

11. r (4)
    × $(7, 11)$

Further example

- all branches are closed
- intuitive meaning: different cases are distinguished, but each case leads to a contradiction
- this disproves the assumption, hence it proves the original tautology

The calculus of truth trees

- procedure can partially be automatized
- every complex formula leads in a well-defined way to an extension of the truth tree

Rules

- double negation

$$(DN) \quad \begin{array}{c} \neg\neg\varphi \\ \hline \varphi \end{array}$$

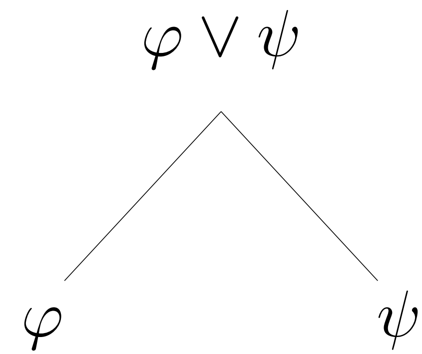
- conjunction

$$(C) \quad \begin{array}{c} \varphi \wedge \psi \\ \hline \varphi \\ \psi \end{array}$$

Rules

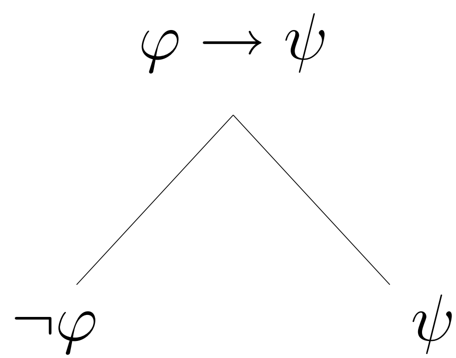
- disjunction

(D)



- implication

(I)



Rules

- equivalence

$$(E) \qquad \varphi \leftrightarrow \psi$$

$$\varphi \quad \psi \qquad | \qquad \neg\varphi \quad \neg\psi$$

- negation + conjunction

$$(NC) \qquad \neg(\varphi \wedge \psi)$$

$$\neg\varphi \qquad | \qquad \neg\psi$$

Rules

- negation + disjunction

$$
\begin{array}{ll}
(ND) & \neg(\varphi \vee \psi) \\
& \neg\varphi \\
& \neg\psi
\end{array}
$$

- negation + implication

$$
\begin{array}{ll}
(NI) & \neg(\varphi \longrightarrow \psi) \\
& \varphi \\
& \neg\psi
\end{array}
$$

Rules

- negation + equivalence

$$(NE) \qquad \neg(\varphi \leftrightarrow \psi)$$

φ	$\neg\varphi$
$\neg\psi$	ψ

Calculus of truth trees

Theorem 7 *A formula* φ *of statement logic is logically true if and only if every branch of a truth tree, starting with* $\neg\varphi$ *as root, that only uses the rules given above, can be closed with an "x" because some formula occurs in it both in negated and in un-negated form.*

Rules of thumb

- Try to use the non-branching rules first.
- When applying a branching rule, try to do it so that one of the two branches can be closed soon.
- The double negation of an atomic statement is usually good for nothing; therefore develop doubly negated atoms only if it is necessary to close a branch.