Mathematics for linguists

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General remarks

- there is no simple algorithm to prove a given theorem/derivation
- you can always start a sub-proof with any arbitrary new hypotheses
- hence there are infinitely many proofs for each derivation
- **but:** it is not possible to prove via natural deduction that a formula is not derivable from a given set of premises
- if you suspect that the conclusion doesn't follow from the premises, it is safer to work with truth trees

Rules of thumb

- always keep track which sub-goal you are currentyl proving
- if the current sub-goal is $\varphi \wedge \psi$:
 - first prove φ
 - then prove ψ
 - then apply $\wedge I$
- if the current sub-goal is $\neg \varphi$:
 - \checkmark start a sub-proof with φ as additional assumption
 - for some convenient formula ψ : prove both ψ and $\neg \psi$
 - finish the sub-proof with $I\neg$

- If the current sub-goal is $\varphi \to \psi$:
 - start a new sub-proof with φ as additional assumption
 - \checkmark try to prove ψ
 - if successful: finish the sub-proof with $\rightarrow I$

- if the current sub-goal is $\varphi \lor \psi$:
 - prove φ or
 - \checkmark prove ψ
 - if successful, introduce $\varphi \lor \psi$ via $\lor I, 1(2)$

• otherwise: if there is an accessible formula $\xi \lor \zeta$

- combine $\lor E$ and $\lor I$:
- start a sub-proof with the assumption ξ and prove φ (or ψ)
- derive $\varphi \lor \psi$ using $\lor I$ and finish sub-proof
- start a second sub-proof and prove ψ (φ)
- \checkmark from this, derive $\varphi \lor \psi$ via $\lor I$ and finish sub-proof
- via $\lor E$, derive $\varphi \lor \psi$

- if the currect sub-goal is $\varphi \leftrightarrow \psi$:
 - \bullet start sub-proof with the additional assumption φ
 - prove ψ
 - In the sub-proof and start new sub-proof withe the assumption ψ
 - prove φ
 - finish the second sub-proof and apply $\leftrightarrow I$

further rules of thumb:

- apply $\wedge E$, $\rightarrow E$ and $\leftrightarrow E$ whenever possible
- Iso, apply ¬I as soon as possible; if the current line in the proof is the negation of an earlier accessible line, immediately end the current sub-proof with ¬I.

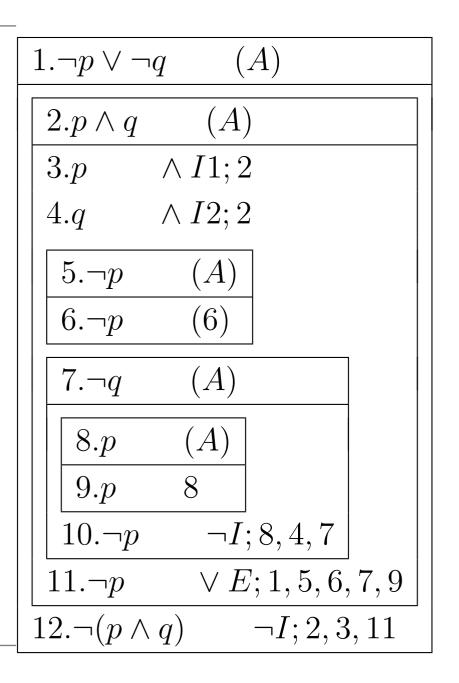
- if none of these rules of thumb is applicable: indirect proof:
- suppose you want to prove φ
 - start your sub-proof with the assumption $\neg \varphi$
 - try to derive a contradiction
 - i.e.: try to derive both ψ and $\neg\psi$ for some formula ψ
 - if successful: end the current sub-proof with $\neg I$
 - result is $\neg \neg \varphi$
 - applying $\neg E$ leads to φ , as desired

Examples: de Morgan's laws (1)

$$\neg (p \land q) \vdash \neg p \lor \neg q$$

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Examples: de Morgan's laws (2)



 $\neg p \lor \neg q \vdash \neg (p \land q)$

Examples: de Morgan's laws (3)

$$\begin{array}{c|cccc}
1.\neg(p \lor q) & (A) \\
\hline
2.p & (A) \\
\hline
3.p \lor q & \lor I1; 2 \\
4.\neg p & \neg I; 2, 1, 3 \\
\hline
5.q & (A) \\
\hline
6.p \lor q & \lor I2; 5 \\
7.\neg q & \neg I; 5, 1, 6 \\
8.\neg p \land \neg q & \land I; 4, 7 \\
\end{array}$$

$$\neg (p \lor q) \vdash \neg p \land \neg q$$

Examples: de Morgan's laws (4)

$1.\neg p \land \neg q$	(A)
$2.\neg p$ \land	XI1;1
$3.\neg q$ \land	12;1
$4.p \lor q$	(A)
5.p (A)
6.p 5	
7.q (.	A)
$ 8.\neg p$	(A)
$ 9.\neg p $	8
$\left \right \left \overline{10.\neg \neg p} \right $	egreen I; 8, 3, 7
11. <i>p</i>	$\neg E;10$
12. <i>p</i>	$\checkmark E; 4, 5, 6, 7, 11$
$\boxed{13.\neg(p\lor q)}$	eg I; 4, 2, 12

$$\neg p \land \neg q \vdash \neg (p \lor q)$$

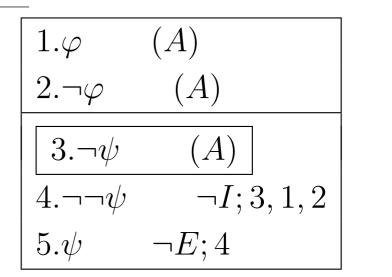
Lemmas

Out rule:

$$\frac{M \Rightarrow \varphi \qquad N, \varphi \Rightarrow \xi}{M, N \Rightarrow \xi}$$

- a derivation, if proved once, can be re-used
- simplifies practical work a lot

Ex falsum quod libet



$$\varphi,\neg\varphi\vdash\psi$$

- once we proved this inference, we can use it as a new rule from now on
- if both φ and $\neg \varphi$ are accessible at some point for any arbitrary formula φ —, you can add any formula you want

Summary: statement logic

- covered here: classical statement logic
- besides, there is a multitude of non-classical statement logics (intuitionistic logic, relevant logic, modal logics, linear logic, ...)

Summary: statement logic

- meta-logical properties of classical statement logic:
 - two-valued semantics (every statement is either true or false)
 - there is a sound and complete syntactic description of logical inference; there are several systems of syntactic rules (truth trees, natural deduction, ...) that identify exactly the set of tautologies
 - logical inference is decidable: there are mechanical decision procedures (for instance truth tables) that distinguish tautologies from non-tautologies

Summary: statement logic

- beyond statement logic:
 - classical first order logic (covered in the remainder of this course) has a sound and complete syntactic proof system, but is not decidable
 - second order logic (and higher order logics) and type theory are neither decidadble, nor do they have a complete syntactic proof system (i.e. it is not possible to describe the set of tautologies by means of finitely many syntactic rules)