Mathematics for linguists

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University of Tübingen

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Gerhard Jäger

Natural deduction

General remarks

- there is no simple algorithm to prove a given theorem/derivaiton
- you can always start a sub-proof with any arbitrary new hypotheses
- hence there are infinitely many proofs for each derivation
- **but:** it is not possible to prove via natural deduction that a formula is not derivable from a given set of premises
- if you suspect that the conclusion doesn't follow from the premises, it is safer to work with truth trees

Natural deduction

Rules of thumb

- always keep track which sub-goal you are currentyl proving
- if the current sub-goal is $\varphi \wedge \psi$:
  - first prove φ
  - then prove ψ
  - then apply $\wedge I$
- if the current sub-goal is $\neg\varphi$:
  - start a sub-proof with φ as additional assumption
  - for some convenient formula ψ: prove both ψ and $\neg\psi$
  - finish the sub-proof with $I\neg$

Natural deduction

- if the current sub-goal is $\varphi \rightarrow \psi$:
  - start a new sub-proof with φ as additional assumption
  - try to prove ψ
  - if successful: finish the sub-proof with $\rightarrow I$

Natural deduction

- if the current sub-goal is $\varphi \vee \psi$:
  - prove φ **or**
  - prove ψ
  - if successful, introduce $\varphi \vee \psi$ via $\vee I, 1(2)$

Natural deduction

- otherwise: if there is an accessible formula $\xi \vee \zeta$
  - combine $\vee E$ and $\vee I$:
  - start a sub-proof with the assumption ξ and prove φ (or ψ)
  - derive $\varphi \vee \psi$ using $\vee I$ and finish sub-proof
  - start a second sub-proof and prove ψ (φ)
  - from this, derive $\varphi \vee \psi$ via $\vee I$ and finish sub-proof
  - via $\vee E$, derive $\varphi \vee \psi$

Natural deduction

- if the currect sub-goal is $\varphi \leftrightarrow \psi$:
  - start sub-proof with the additional assumption φ
  - prove ψ
  - finish sub-proof and start new sub-proof withe the assumption ψ
  - prove φ
  - finish the second sub-proof and apply $\leftrightarrow I$

Natural deduction

- further rules of thumb:
  - apply $\wedge E$, $\rightarrow E$ and $\leftrightarrow E$ whenever possible
  - also, apply $\neg I$ as soon as possible; if the current line in the proof is the negation of an earlier accessible line, immediately end the current sub-proof with $\neg I$.

Natural deduction

- if none of these rules of thumb is applicable: **indirect proof**:
- suppose you want to prove φ
  - start your sub-proof with the assumption $\neg\varphi$
  - try to derive a contradiction
  - i.e.: try to derive both ψ and $\neg\psi$ for some formula ψ
  - if successful: end the current sub-proof with $\neg I$
  - result is $\neg\neg\varphi$
  - applying $\neg E$ leads to φ, as desired

Examples: de Morgan's laws (1)

$$
\begin{array}{|l|}
\hline
1. \neg(p \wedge q) \qquad (A) \\
\hline
\begin{array}{|l|}
\hline
2. \neg(\neg p \vee \neg q) \qquad (A) \\
\hline
\begin{array}{|l|}
\hline
3. \neg p \qquad (A) \\
\hline
4. \neg p \vee \neg q \qquad \vee I 1; 3 \\
\hline
\end{array} \\
5. \neg\neg p \qquad \neg I; 3, 4, 2 \\
\begin{array}{|l|}
\hline
6. \neg q \qquad (A) \\
\hline
7. \neg p \vee \neg q \qquad \vee I 2; 6 \\
\hline
\end{array} \\
8. \neg\neg q \qquad \neg I; 6, 7, 2 \\
9. p \qquad \neg E; 5 \\
10. q \qquad \neg E; 8 \\
11. p \wedge q \qquad \wedge I; 9, 10 \\
\hline
\end{array} \\
12. \neg\neg(\neg p \vee \neg q) \qquad \neg I; 2, 11, 1 \\
13. \neg p \vee \neg q \qquad \neg E; 12 \\
\hline
\end{array}
$$

$$\neg(p \wedge q) \vdash \neg p \vee \neg q$$

Examples: de Morgan's laws (2)

1. $\neg p \vee \neg q$	(A)
2. $p \wedge q$	(A)
3. p	$\wedge I 1; 2$
4. q	$\wedge I 2; 2$
5. $\neg p$	(A)
6. $\neg p$	(6)
7. $\neg q$	(A)
8. p	(A)
9. p	8
10. $\neg p$	$\neg I; 8, 4, 7$
11. $\neg p$	$\vee E; 1, 5, 6, 7, 9$
12. $\neg(p \wedge q)$	$\neg I; 2, 3, 11$

$$\neg p \vee \neg q \vdash \neg(p \wedge q)$$

Examples: de Morgan's laws (3)

$1. \neg(p \vee q)$	(A)
$2. p$	(A)
$3. p \vee q$	$\vee I 1; 2$
$4. \neg p$	$\neg I; 2, 1, 3$
$5. q$	(A)
$6. p \vee q$	$\vee I 2; 5$
$7. \neg q$	$\neg I; 5, 1, 6$
$8. \neg p \wedge \neg q$	$\wedge I; 4, 7$

$$\neg(p \vee q) \vdash \neg p \wedge \neg q$$

Examples: de Morgan's laws (4)

$1. \neg p \wedge \neg q$	(A)
$2. \neg p$	$\wedge I1; 1$
$3. \neg q$	$\wedge I2; 1$
$4. p \vee q$	(A)
$5. p$	(A)
$6. p$	5
$7. q$	(A)
$8. \neg p$	(A)
$9. \neg p$	8
$10. \neg\neg p$	$\neg I; 8, 3, 7$
$11. p$	$\neg E; 10$
$12. p$	$\vee E; 4, 5, 6, 7, 11$
$13. \neg(p \vee q)$	$\neg I; 4, 2, 12$

$$\neg p \wedge \neg q \vdash \neg(p \vee q)$$

Lemmas

- Cut rule:

$$\frac{M \Rightarrow \varphi \qquad N, \varphi \Rightarrow \xi}{M, N \Rightarrow \xi}$$

- a derivation, if proved once, can be re-used
- simplifies practical work a lot

Ex falsum quod libet

$1. \varphi$	(A)
$2. \neg\varphi$	(A)
$3. \neg\psi$	(A)
$4. \neg\neg\psi$	$\neg I; 3, 1, 2$
$5. \psi$	$\neg E; 4$

$$\varphi, \neg\varphi \vdash \psi$$

- once we proved this inference, we can use it as a new rule from now on
- if both φ and $\neg\varphi$ are accessible at some point — for any arbitrary formula φ —, you can add any formula you want

Summary: statement logic

- covered here: **classical statement logic**
- besides, there is a multitude of non-classical statement logics (intuitionistic logic, relevant logic, modal logics, linear logic, ...)

Summary: statement logic

- meta-logical properties of classical statement logic:
  - two-valued semantics (every statement is either true or false)
  - there is a sound and complete syntactic description of logical inference; there are several systems of syntactic rules (truth trees, natural deduction, ...) that identify exactly the set of tautologies
  - logical inference is **decidable**: there are mechanical decision procedures (for instance truth tables) that distinguish tautologies from non-tautologies

Summary: statement logic

- beyond statement logic:
  - classical first order logic (covered in the remainder of this course) has a sound and complete syntactic proof system, but is not decidable
  - second order logic (and higher order logics) and type theory are neither decidadble, nor do they have a complete syntactic proof system (i.e. it is not possible to describe the set of tautologies by means of finitely many syntactic rules)