Mathematics for linguists

Gerhard Jäger
gerhard.jaeger@uni-tuebingen.de

Uni Tübingen, WS 2009/2010

November 5, 2009

Orderings

- distinction between **strict** (or strong) and **weak orders**

Weak orderings

A relation R is a **weak ordering** iff R is

- transitive,
- reflexive, and
- anti-symmetric

Strict ordering

A relation R is a **strict orderings** iff R is

- transitive,
- irreflexive, and
- asymmetric

Orderings

Examples:

- $A = \{a, b, c, d\}$
- $R_1 = \{\langle a, b\rangle, \langle a, c\rangle, \langle a, d\rangle, \langle b, c\rangle, \langle a, a\rangle, \langle b, b\rangle, \langle c, c\rangle, \langle d, d\rangle\}$
- $R_2 = \{\langle b, a\rangle, \langle b, b\rangle, \langle a, a\rangle, \langle c, c\rangle, \langle d, d\rangle, \langle c, b\rangle, \langle c, a\rangle\}$
- $R_3 =$
  $\{\langle d, c\rangle, \langle d, b\rangle, \langle d, a\rangle, \langle c, b\rangle, \langle c, a\rangle, \langle a, a\rangle, \langle b, b\rangle, \langle c, c\rangle, \langle d, d\rangle, \langle b, a\rangle\}$

corresponding strict orderings:

- $S_1 = \{\langle a, b\rangle, \langle a, c\rangle, \langle a, d\rangle, \langle b, c\rangle\}$
- $S_2 = \{\langle b, a\rangle, \langle c, b\rangle, \langle c, a\rangle\}$
- $S_3 = \{\langle d, c\rangle, \langle d, b\rangle, \langle d, a\rangle, \langle c, b\rangle, \langle c, a\rangle, \langle b, a\rangle\}$

Orderings

A weak ordering $R \subseteq A \times A$ and a strict ordering S correspond to each other iff

$$R = S \cup id_A$$

- further examples:
  - $\leq$ and $< \subseteq \mathbb{N} \times \mathbb{N}$
  - $\subseteq$ and $\subset$ are subsets of $\wp(A) \times \wp(A)$

Orderings

Terminology

Let R be an ordering (strict or weak).

- a is a *predecessor* of b iff $R(a, b)$.
- a is a *successor* of b iff $R(b, a)$.
- a is a *immediate predecessor* of b iff
  - $a \neq b$,
  - $R(a, b)$, and
  - there is no $c \notin \{a, b\}$ such that $R(a, c)$ and $R(c, b)$.
- a is an *immediate successor* of b iff b is an immediate predecessor of a.

Orderings

More terminology

Let $R \subseteq A \times A$ be an ordering (strict or weak).

- An element $x \in A$ is *minimal* iff there is no $y \neq x$ which is a predecessor of x.
- An element $x \in A$ is *least* iff x is the predecessor of all other elements of A.
- An element $x \in A$ is *maximal* iff there is no $y \neq x$ which is a successor of x.
- An element $x \in A$ is *greatest* iff x is the sucessor of all other elements of A.

An ordering has at most one least and at most one greatest element, but ther may be any number of minimal or maximal elements. The least element is always minimal, and the greatest element is always maximal.

Orderings

Linear ordering

An ordering is **linear** (or **total**) iff it is connected.

If an ordering R is not linear, there are two elements a and b such that neither $R(a, b)$ nor $R(b, a)$. The ordering is thus not complete, but partial.

Orderings

- further kinds of orderings:
  - An ordering $R \subseteq A \times A$ is *well-founded* iff every restriction $R \cap (B \times B)$ to a subset $B \subset A$ contains a minimal element. To put it another way: a well-founded ordering does not have infinitely descending branches.
  - An ordering $R \subseteq A \times A$ is *dense* iff for any pair of elements x and y with $R(x, y)$, there is a third element z, which differs from x and y, such that $R(x, z)$ and $R(z, y)$.