Mathematics for linguists

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Ordered pairs

- sets are not ordered: $\{a, b\} = \{b, a\}$
- for many applications we need ordered structures
- most basic example: ordered pair $\langle a, b\rangle$
  - ordered:

    If $a \neq b$, then $\langle a, b\rangle \neq \langle b, a\rangle$.

  - extensional:

    $\langle a_1, b_1\rangle = \langle a_2, b_2\rangle$ if and only if $a_1 = a_2$ and $b_1 = b_2$.

Set theoretic definition

$$\langle a, b\rangle \doteq \{\{a\}, \{a, b\}\}$$

Ordered pairs and tuples

- set theoretic definition does what it is supposed to do, because:
  - If $a \neq b$, then $\{\{a\},\{a,b\}\} \neq \{\{a\},\{a,b\}\}$.
  - $\{\{a_1\},\{a_1,b_1\}\} = \{\{a_2\},\{a_2,b_2\}\}$ if and only if $a_1 = a_2$ and $b_1 = b_2$.
- ordered n-tuples can be defined recursively as ordered pairs

$$\begin{aligned}
\langle a,b,c\rangle &\doteq \langle\langle a,b\rangle, c\rangle \\
\langle a,b,c,d\rangle &\doteq \langle\langle a,b,c\rangle, d\rangle \\
&\vdots \\
\langle a_1,\dots,a_n\rangle &= \langle\langle a_1,\dots,a_{n-1}\rangle, a_n\rangle
\end{aligned}$$

The Cartesian product

- Cartesian product:
  - operation between two sets
  - notation: $A \times B$
  - set of all ordered pairs, such that the first element comes from A and the second one from B:

$$A \times B = \{\langle a, b\rangle | a \in A \text{ and } b \in B\}$$

The Cartesian product

- examples
  - Let $K = \{a, b, c\}$ and $L = \{1, 2\}$.

$$\begin{aligned}
K \times L &= \{\langle a,1\rangle, \langle a,2\rangle, \langle b,1\rangle, \langle a,2\rangle, \langle c,1\rangle, \langle c,2\rangle\} \\
L \times K &= \{\langle 1,a\rangle, \langle 1,b\rangle, \langle 1,c\rangle, \langle 2,a\rangle, \langle 2,b\rangle, \langle 2,c\rangle\} \\
K \times K &= \{\langle a,a\rangle, \langle a,b\rangle, \langle a,c\rangle, \langle b,a\rangle, \langle b,b\rangle, \langle b,c\rangle, \\
&\quad \langle c,a\rangle, \langle c,b\rangle, \langle c,c\rangle\} \\
L \times L &= \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,1\rangle, \langle 2,2\rangle\} \\
K \times \emptyset &= \emptyset \\
L \times \emptyset &= \emptyset
\end{aligned}$$

Observation: If A and B are finite, then:

$$|A \times B| = |A| \times |B|$$

The Cartesian product

- Cartesian product between more than two sets:
  - $A \times B \times C \doteq (A \times B) \times C$
  - similarly for more than three sets
  - $A \times B \times C$ is the set of all triples ("3-tuple"), such that the first component is an element of A, the second one an element of B, the the third one an element of C.
  - again, this holds analogously for more than three sets
- Notations:
  - $\Pi_{1 \leq i \leq n} A_i \doteq A_1 \times A_2 \times \cdots \times A_n$ (Do not confuse with projection operations!)
  - $A^n \doteq \underbrace{A \times \cdots \times A}_{n \text{ times}}$

Projections

- projection operations map an ordered pair to on of its components:

$$\begin{aligned}\pi_0(\langle a, b\rangle) &\doteq a\\ \pi_1(\langle a, b\rangle) &\doteq b\end{aligned}$$

- Besides, there are projection operations from sets of ordered pairs to the set of the first (second) elements:

$$\begin{aligned}\Pi_0(R) &\doteq \{x|\text{There is an } a \in R \text{ such that } \pi_0(a) = x\}\\ \Pi_1(R) &\doteq \{x|\text{There is an } a \in R \text{ such that } \pi_1(a) = x\}\end{aligned}$$

Relations

- Intuitive basis:
  - A (binary) relation is a relation between two objects.
  - Can be expressed by a transitive verb or a construction like *[noun] of/[adjective in comparative form] than*
  - examples:
    - mother of
    - taller than
    - predecessor of
    - loves
    - is interested in
    - ...

Relations

- mathematical modeling: **extensional**
- It is only important between **which objects** a relation holds; it is not important **how** the relation is characterized
- for instance: If every person (within the universe of discourse) loves their spouse and nobody loves anybody else than their spouse, then the relations of “loving” and of “is spouse of” are identical.

Relations

- notation:
  - relations are frequently written as $R, S, T, \ldots$
  - "a stands in relation R to b" is written as $R(a,b)$ or Rab or aRb
- A relation is a set of ordered pairs.

Definition

R is a relation iff there are sets A and B such that $R \subseteq A \times B$.

The notation Rab $(R(a,b), aRb)$ is thus a shorthand for $\langle a, b\rangle \in R$.

Relations

Let $R \subseteq A \times B$.

- R is a relation between A and B or from A to B.
- $\pi_0[R] := \{a \in A | a = \pi_0(\langle a, b \rangle) \text{ for some } \langle a, b \rangle \in R\} \subseteq A$
- $\pi_1[R] := \{b \in B | b = \pi_1(\langle a, b \rangle) \text{ for some } \langle a, b \rangle \in R\} \subseteq B$
- $\pi_0[R]$ is the domain of R (German: *Definitionsbereich*)
- $\pi_1[R]$ is the Range of R (German: *Wertebereich*)

Relations are sets, hence set theoretic operations are defined for them. For instance:

$$\overline{R} = (A \times B) - R$$

Inverse relation

Let $R \subseteq A \times B$.

- R^{-1} is the inverse Relation to R.
- Rab iff $R^{-1}ba$
- $R^{-1} := \{\langle a, b\rangle \in B \times A | \langle b, a\rangle \in R\}$
- $\pi_0[R] = \pi_1[R^{-1}]$
- $\pi_1[R] = \pi_0[R^{-1}]$

Relations

Examples:

- $A = \{1, 2, 3\}$
- $B = \{a, b, c\}$
- $R = \{\langle 1, a\rangle, \langle 1, c\rangle, \langle 2, a\rangle\}$
- $\pi_0[R] = \{1, 2\} \subseteq A$
- $\pi_1[R] = \{a, c\} \subseteq B$
- $\overline{R} = \{\langle 1, b\rangle, \langle 2, b\rangle, \langle 2, c\rangle, \langle 3, a\rangle, \langle 3, b\rangle, \langle 3, c\rangle\}$
- $R^{-1} = \{\langle a, 1\rangle, \langle c, 1\rangle, \langle a, 2\rangle\}$

Relations

- notion of a relation can be generalized to dependencies of higher arity
- examples for ternary relations: "between", "are parents of", ...
- formally: an n-ary relation is a set of n-tuples
- $R \subseteq A_1 \times \cdots \times A_n$

Functions

- functions: special kind of relations
- Let $f \subseteq A \times B$ be a relation between A and B. f is a **function** iff every element of $\pi_0[f]$ is paired with exactly one element of B.
- $f \subseteq A \times B$ is a function from A to B iff $\pi_0[f] = A$.

examples:

- $A = \{a, b, c\}$ and $B = \{1, 2, 3, 4\}$
- functions:

$$\begin{array}{rcl} P & = & \{\langle a, 1\rangle, \langle b, 2\rangle, \langle c, 3\rangle\} \\ Q & = & \{\langle a, 3\rangle, \langle b, 4\rangle, \langle c, 1\rangle\} \\ R & = & \{\langle a, 3\rangle, \langle b, 2\rangle, \langle c, 2\rangle\} \end{array}$$

- no functions:

$$\begin{array}{rcl} S & = & \{\langle a, 1\rangle, \langle b, 2\rangle\} \\ T & = & \{\langle a, 2\rangle, \langle b, 3\rangle, \langle a, 3\rangle, \langle c, 1\rangle\} \\ V & = & \{\langle a, 2\rangle, \langle a, 3\rangle, \langle b, 4\rangle\} \end{array}$$

Functions

- notations and writing conventions:
  - we frequently used the letters f, g, F, G, H etc. for functions
  - $f : A \to B$ means "f is a function, $f \subseteq A \times B$ and $\pi_0[f] = A$"
  - $f(a) = b$ (or also: $f : a \mapsto b$) is shorthand for "$\langle a, b\rangle \in f$"
  - elements of the domain are called arguments of the function
  - elements of the range are called values of the function
  - f is called surjective (or "onto") iff every element of B is paired with at least one argument, i.e. $\pi_1[f] = B$.
  - f is called injective (or "1-1") if every element of B is paired with at most one argument.
  - f is called bijective (oder "1-1 onto"), if it is injective and surjective.

The function f is bijective iff f^{-1} is also a function. In this case, f^{-1} is called the inverse function of f.

Functions

- Functions are frequently defined via some rule that enables us to find the value for each argument.
- examples:
  - $f(x) = x + 2$
  - $g(x) = x^2$
  - $h(x) = 3x^2 + 2x + 1$
- To decide which functions are defined here, we need to know the domain and the range.
- Question: Under what conditions do these rules define injective, surjective and/or bijective functions?

Functions of higher arity

- Domain of a function may be a relation
- examples:
  - $A = \{1, 2\}$, $B = \{a, b\}$, $C = \{\alpha, \beta\}$
  - $F : A \times B \to C$
  - $F = \{\langle 1, a, \alpha\rangle, \langle 1, b, \alpha\rangle, \langle 2, a, \beta\rangle, \langle 2, b, \alpha\rangle\}$
- Instead of $F(\langle 1, a\rangle)$ etc. we usually write $F(1, a)$ etc.
- If the domain of a function is an n-ary relation, we speak of an n-ary function.
- Note: n-ary functions are $n + 1$-ary relations!