Mathematics for linguists

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October 21, 2010

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Translation English $\Rightarrow$ statement logic

- motivation for translation:
  1. English as object-language: translation admits modeling of the semantics of English using the means of logic
  2. English as meta-language: translation helps to make the notion of the valid argument precise

A statement A is an adequate translation of a statement A' if and only if A and A' have the same truth conditions.

Translation

- translation of an English statement A consists of
  - a statement A' of statement logic, and
  - conditions for the valuation V of statement logic
- a good translation of A is
  - as poor in structure as possible, and
  - as similar in structure as possible to A

Translation: negation

Example

- English:

  (1) Paul is not smart.

- translation:

  (2) a. $\neg p$

  b. p : Paul is smart.

- rule of thumb: *If an English statement that contains "not" (or "n't") can be paraphrased without problems by a formulation using "it is not the case that", then A can be translated into a negated formula.*

Translation: negation

paraphrase test is also useful for other English expressions for negation:

- English:

  (3) Franz Beckenbauer owns *no* cars.

- paraphrase:

  (4) *It is not the case that* Franz Beckenbauer owns a car.

- translation:

  (5) a. $\neg p$

  b. p : Franz Beckenbauer owns a car.

Translation: negation

- Further examples:

(6) a. *Nobody* is smarter than John.
  b. It is not the case that somebody is smarter than John.
  c. $\neg p/p$: Somebody is smarter than John.

(7) a. Fritz donated *nothing*.
  b. It is not the case that Fritz donated something.
  c. $\neg p/p$: Fritz donated something.

(8) a. *Neither* John *nor* Peter are in Tübingen.
  b. It is not the case that John or Peter is in Tübingen.
  c. $\neg p/p$: John or Peter is in Tübingen.

Translation: negation

(9) a. John is *un*reasonable.
b. It is not the case that John is reasonable.
c. $\neg p/p$: John is reasonable.

but:

(10) a. John unloads the truck.
b. $\neq$ It is not the case that John loads the truck.
c. (correct translation:) p/p : John unloads the truck.

Translation: conjunction

(11)
a. John is blond and John is six feet tall.
b. $p \wedge q$
c. p : John is blond.
d. q : John is six feet tall.

(12)
a. John is blond and six feet tall.
b. (paraphrase:) John is blond and John is six feet tall.
c. $p \wedge q$
d. p : John is blond.
e. q : John is six feet tall.

Translation: conjunction

(13)
a. John and Paul are good swimmers.
b. John is a good swimmer and Paul is a good swimmer.
c. $p \wedge q$
d. p : John is a good swimmer. q : Paul is a good swimmer.

- rule of thumb: *If a statement* A *that contains "and" can be paraphrased by a sentence where "and" connects two clauses, then* A *can be translated as a conjunction.*

Translation: conjunction

but:

(14)
a. John and Gerda are married.
b. $\neq$ John is married and Gerda is married.
c. (correct translation:) p
d. p : John and Gerda are married.

Translation: conjunction

- further ways to express conjunctive statements:

(15) a. John is *both* stupid *and* lazy.
b. John is stupid and John is lazy.
c. $p \wedge q$
d. p : John is stupid. q : John is lazy.

(16) a. John is not stupid, *but* he is lazy.
b. John is not stupid and John is lazy.
c. $\neg p \wedge q$
d. p : John is stupid. q : John is lazy.

Translation: conjunction

(17)
a. *Even though* Helga is engaged to Paul, she does not love him.
b. Helga is engaged to Paul, and Helga does not love Paul.
c. $p \wedge \neg q$
d. p : Helga is engaged to Paul. q : Helga loves Paul.

Translation: disjunction

- regarding the problem of exclusive vs. inclusive reading of “or”: see last lecture
- apart from that, disjunction relates to “or” as conjunction to “and”

(18)
a. John is blond or John is six feet tall.
b. $p \vee q$
c. p : John is blond.
d. q : John is six feet tall.

Translation: disjunction

(19)
a. John is blond or six feet tall.
b. (paraphrase:) John is blond or John is six feet tall.
c. $p \vee q$
d. p : John is blond.
e. q : John is six feet tall.

(20)
a. John or Paul is a good swimmer.
b. John is a good swimmer or Paul is a good swimmer.
c. $p \vee q$
d. p : John is a good swimmer. q : Paul is a good swimmer.

Translation: implication

- There is no real counterpart to implication in English.
- Some grammatical constructions can approximately translated by implications.
- rule of thumb: *Suppose* A *is an English statement which might possibly be translated as an implication* $\varphi \rightarrow \psi$*. To test the adequacy of this translation, it is important to understand under what conditions* A *is false. If the translation is correct, then under these very conditions,* φ *must be true and* ψ *false.*

Translation: implication

(21)
a. *If* John is the father of Paul, *then* John is older than Paul.
b. $p \to q$
c. p : John is the father of Paul.
d. q : John is older than Paul.

(22)
a. John will come to the party *only if* Helga comes.
b. $p \to q$
c. p : John will come to the party.
d. q : Helga will come to the party.

Translation: implication

(23)
a. That x is even is a *necessary condition* that x is divisible by 4.
b. $p \rightarrow q$
c. p : x is divisible .
d. q : x is even.

(24)
a. That x is divisible by 4 is a *sufficient condition* that x is even.
b. $p \rightarrow q$
c. p : x is divisible by 4.
d. q : x is even.

Translation: Equivalence

(25) a. John comes to the party if and only if Paul comes.
b. $p \leftrightarrow q$
c. p : John comes to the party.
d. q : Paul comes to the party.

(26) a. John comes to the party just in case Paul comes.
b. $p \leftrightarrow q$
c. p : John comes to the party.
d. q : Paul comes to the party.

Translation: equivalence

(27) a. That the last digit in the decimal representation of x is 0 is a *necessary and sufficient condition* that x is divisible by 10.

b. $p \leftrightarrow q$

c. p : The last digit in the decimal representation of x is 0.

d. q : x is divisible by 10.

Tautologies

Definition (Tautology)

A formula of statement logic φ is a **tautology** *of statement logic, formally written as*

$$\Rightarrow \varphi$$

if and only if it holds for all valuations V:

$$V(\varphi) = 1$$

Tautologies

- Tautologies are called *logically true*.
- Examples for tautologies:

$$p \vee \neg p, \neg(p \wedge \neg p), p \rightarrow q \rightarrow p, p \rightarrow \neg\neg p, p \rightarrow p \vee q, ...$$

- Whether or not a formula is logically true can be decided with the help of truth tables. Logically true formulas are true under each valuation function, i.e. in each row.

Tautologies

	p	q	$q \to p$	$p \to q \to p$
V_1	1	1		
V_2	1	0		
V_3	0	1		
V_4	0	0		

Tautologies

	p	q	$q \to p$	$p \to q \to p$
V_1	1	1	1	
V_2	1	0	1	
V_3	0	1	0	
V_4	0	0	1	

Tautologies

	p	q	$q \rightarrow p$	$p \rightarrow q \rightarrow p$
V_1	1	1	1	1
V_2	1	0	1	1
V_3	0	1	0	1
V_4	0	0	1	1

Contradictions

Definition (Contradiction)

A formula φ *is a* **contradiction** *of statement logic if and only if it holds for all valuation functions* V*:*

$$V(\varphi) = 0$$

- Contradictions are called *logically false*.
- Examples for contradictions:

$$p \wedge \neg p, \neg(p \vee \neg p), (p \to \neg p) \wedge p, p \leftrightarrow \neg p, ...$$

- Whether or not a formula is logically false can also be determined by using truth tables. Logically false formulas are false under each valuation function, i.e. in each row.

Contradictions

	p	$\neg p$	$p \rightarrow \neg p$	$(p \rightarrow \neg p) \wedge p$
V_1	1			
V_2	0			

Contradictions

	p	$\neg p$	$p \rightarrow \neg p$	$(p \rightarrow \neg p) \wedge p$
V_1	1	0		
V_2	0	1		

Contradictions

	p	$\neg p$	$p \rightarrow \neg p$	$(p \rightarrow \neg p) \wedge p$
V_1	1	0	0	
V_2	0	1	1	

Contradictions

	p	$\neg p$	$p \rightarrow \neg p$	$(p \rightarrow \neg p) \wedge p$
V_1	1	0	0	0
V_2	0	1	1	0

Tautologies and contradictions

Theorem

If φ is a tautology, then $\neg\varphi$ is a contradiction.

Proof: Suppose the premise is correct and φ is a tautology. Let V be an arbitrary valuation function. By assumption, it holds that

$$V(\varphi) = 1$$

From this it follows that

$$V(\neg\varphi) = 0$$

due to the semantics of negation. Since we did not make any specific assumption about V, it holds for any V that $V(\neg\varphi) = 0$. Hence, by definition, $\neg\varphi$ is a contradiction. $\dashv$

Tautologies and contradictions

Theorem

If φ is a contradiciton, then $\neg\varphi$ is a tautology.

Proof: Suppose the premise is correct and φ is a contradiction. Let V be an arbitrary valuation function. By assumption, it holds that

$$V(\varphi) = 0$$

From this it follows that

$$V(\neg\varphi) = 1$$

due to the semantics of negation. Since we did not make any specific assumption about V, it holds for any V that $V(\neg\varphi) = 0$. Hence, by definition, $\neg\varphi$ is a tautology. $\dashv$

Logical equivalence

Definition (Logical equivalence)

Two formulas φ *and* ψ *are* **logically equivalent**, *formally written as*

$$\varphi \Leftrightarrow \psi$$

if and only if for all valuation functions V *it holds that:*

$$V(\varphi) = V(\psi)$$

- Note: "**Logical** equivalence" is a meta-linguistic notion, while "equivalence" in the sense of $\leftrightarrow$ is an operator of the object language.
- Logical equivalence can be decided with the help of truth tables as well.

Logical equivalence

	p	q	r	$p \wedge q$	$q \wedge r$	$p \wedge (q \wedge r)$	$(p \wedge q) \wedge r$
V_1	1	1	1				
V_2	1	1	0				
V_3	1	0	1				
V_4	1	0	0				
V_5	0	1	1				
V_6	0	1	0				
V_7	0	0	1				
V_8	0	0	0				

Hence:

$$(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$$

Logical equivalence

	p	q	r	$p \wedge q$	$q \wedge r$	$p \wedge (q \wedge r)$	$(p \wedge q) \wedge r$
V_1	1	1	1	1			
V_2	1	1	0	1			
V_3	1	0	1	0			
V_4	1	0	0	0			
V_5	0	1	1	0			
V_6	0	1	0	0			
V_7	0	0	1	0			
V_8	0	0	0	0			

Hence:

$$(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$$

Logical equivalence

	p	q	r	$p \wedge q$	$q \wedge r$	$p \wedge (q \wedge r)$	$(p \wedge q) \wedge r$
V_1	1	1	1	1	1		
V_2	1	1	0	1	0		
V_3	1	0	1	0	0		
V_4	1	0	0	0	0		
V_5	0	1	1	0	1		
V_6	0	1	0	0	0		
V_7	0	0	1	0	0		
V_8	0	0	0	0	0		

Hence:

$$(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$$

Logical equivalence

	p	q	r	$p \wedge q$	$q \wedge r$	$p \wedge (q \wedge r)$	$(p \wedge q) \wedge r$
V_1	1	1	1	1	1	1	
V_2	1	1	0	1	0	0	
V_3	1	0	1	0	0	0	
V_4	1	0	0	0	0	0	
V_5	0	1	1	0	1	0	
V_6	0	1	0	0	0	0	
V_7	0	0	1	0	0	0	
V_8	0	0	0	0	0	0	

Hence:

$$(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$$

Logical equivalence

	p	q	r	$p \wedge q$	$q \wedge r$	$p \wedge (q \wedge r)$	$(p \wedge q) \wedge r$
V_1	1	1	1	1	1	1	1
V_2	1	1	0	1	0	0	0
V_3	1	0	1	0	0	0	0
V_4	1	0	0	0	0	0	0
V_5	0	1	1	0	1	0	0
V_6	0	1	0	0	0	0	0
V_7	0	0	1	0	0	0	0
V_8	0	0	0	0	0	0	0

Hence:

$$(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$$

Logical equivalence

Theorem

φ and ψ are logically equivalent if and only if $\varphi \leftrightarrow \psi$ is a tautology.

Proof:

- *Forward direction:* Suppose $\varphi \Leftrightarrow \psi$. Let V be an arbitrary valuation function. By assumption, it holds that $V(\varphi) = V(\psi)$. Hence either $V(\varphi) = V(\psi) = 0$ or $V(\varphi) = V(\psi) = 1$. In either case, it follows from the semantics of the equivalence that $V(\varphi \leftrightarrow \psi) = 1$. $\dashv$

Logical equivalence

- *Backward direction:* Suppose $\varphi \leftrightarrow \psi$ is a tautology. Let V be an arbitrary valuation function. We have to distinguish two cases:
  - $V(\varphi) = 1$. It follows from the semantics of equivalence that $V(\psi) = 1$.
  - $V(\varphi) = 0$. It follows from the semantics of equivalence that $V(\psi) = 0$.

  In both cases it holds that $V(\varphi) = V(\psi)$. Hence φ and ψ are logically equivalent. $\dashv$