Mathematics for linguists

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Inferences and truth trees

- Inferences (with a finite set of premises; from now on we tacitly assume that premise sets are finite) can always be tranformed into tautologies using the deduction theorem
- Inferences can also directly be proved using truth trees though:
  - premises are assumed to be true
  - conclusion is assumed to be false

Inferences and truth trees

- to prove the inference

$$\varphi_1, \ldots, \varphi_n \Rightarrow \psi,$$

start your truth tree with

$$\begin{array}{c} \varphi_1 \\ \vdots \\ \varphi_n \\ \neg\psi \end{array}$$

Inferences and truth trees

Theorem

Let $\varphi_1, \ldots, \varphi_n$ be formulas of statement logic. ψ follows logically from the premises $\varphi_1, \ldots, \varphi_n$ if every branch of a truth tree which starts with $\varphi_1, \ldots, \varphi_n$ and ψ and only uses the known rules, can be closed with an "x" because every formula occurs in it both in negated and non-negated form.

Example

$$p \to q, \neg q \Rightarrow \neg p$$

1. $p \to q$ (A)
2. $\neg q$ (A)
3. $\neg\neg p$ (A)

4.	$\neg p$	(1)	5.	q	(1)
	×	$(3, 4)$		×	$(2, 5)$

Example

- Inference

$$p \to q, p \vee r, \neg r \Rightarrow p \wedge q$$

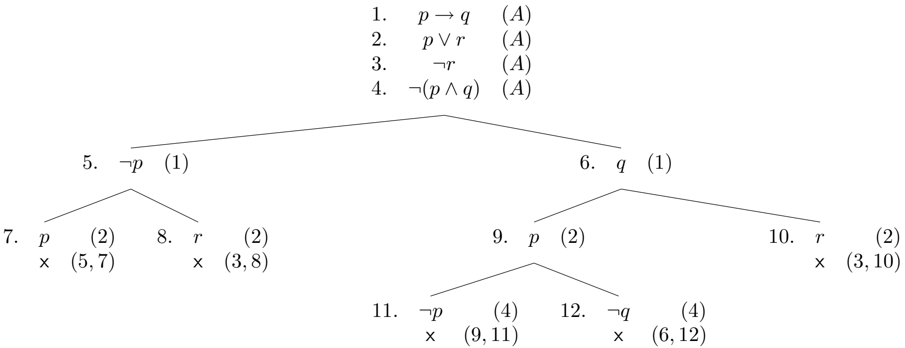
- there is more than one way to prove this

Example

1.	$p \to q$	(A)
2.	$p \lor r$	(A)
3.	$\neg r$	(A)
4.	$\neg(p \land q)$	(A)

- 5. p (2)
  - 7. $\neg p$ (1)
    - × (5, 7)
  - 8. q (1)
    - 9. $\neg p$ (4)
      - × (5, 9)
    - 10. $\neg q$ (4)
      - × (8, 10)
- 6. r (2)
  - × (2, 6)

Example


1. p → q (A)
2. p ∨ r (A)
3. ¬r (A)
4. ¬(p ∧ q) (A)
5. ¬p (1)
6. q (1)
7. p (2)
× (5, 7)
8. r (2)
× (3, 8)
9. p (2)
10. r (2)
× (3, 10)
11. ¬p (4)
× (9, 11)
12. ¬q (4)
× (6, 12)

Natural deduction: motivation

- proving theorems via truth trees is sometimes tedious
- intuitive content of the operators of statement logic is not directly transparent
- for instance, some inferences are obvious from this intutive content:

$$\begin{aligned} \varphi, \psi &\Rightarrow \varphi \wedge \psi \\ \varphi \wedge \psi &\Rightarrow \varphi \\ \varphi, \varphi \rightarrow \psi &\Rightarrow \psi \\ \varphi \rightarrow \psi, \psi \rightarrow \varphi &\Rightarrow \varphi \leftrightarrow \psi \\ &\vdots \end{aligned}$$

Natural deduction: motivation

- meta-logical properties of the inference relation cannot be used
  - identity:

$$\varphi \Rightarrow \varphi$$

  - cut:

$$\frac{M \Rightarrow \varphi \qquad N, \varphi \Rightarrow \xi}{M, N \Rightarrow \xi}$$

  - monotonicity:

$$\frac{M \Rightarrow \varphi}{M, \psi \Rightarrow \varphi}$$

Natural deduction: motivation

- **Calculus of natural deduction:**
  - *syntactic* calculus: only the syntactic form of the formula matters (so the calculus of truth trees is also syntactic, despite its name)
  - two central issues for each operator O:
    - When is is possible to use O in the conclusion of an inference? (introduction rule)
    - What can I do with a premise that contains O as main functor? (elimination rule)

Natural deduction: motivation

- Examples for introduction rules:

$$\frac{M \Rightarrow \varphi \qquad M \Rightarrow \psi}{M \Rightarrow \varphi \wedge \psi}$$

$$\frac{M, \varphi \Rightarrow \psi}{M \Rightarrow \varphi \rightarrow \psi}$$

- Examples for elimination rules

$$\frac{M \Rightarrow \varphi \wedge \psi}{M \Rightarrow \varphi}$$

$$\frac{M \Rightarrow \varphi \rightarrow \psi \qquad M \Rightarrow \varphi}{M \Rightarrow \psi}$$

Calculus of natural deduction

- Notation: we use $\vdash$ (rather than $\Rightarrow$) for syntactically derived inferences
- Terminology:
  - syntactically proven formulas are called **theorems** (which is the *counterpart to the semantic notion of a* **tautology**)
  - If the conclusion φ can be syntactically derived from the premises M, then φ is **derivable** from M *(counterpart to the semantic notion "***follows logically***")*

Natural deduction

- basic structure of a proof (in the calculus of natural deduction):

premises
intermediate steps
⋮
intermediate steps
conclusion

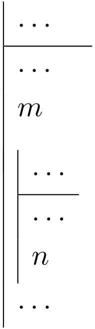
Natural deduction

- **intermediate steps** are
  - formulas that can be derived from preceding lines (within the same sub-proof or within including sub-proofs) by applying an introduction rule or an elimination rule, or
  - complete proofs
  - copies of preceding lines

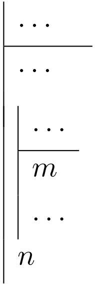
Accessibility

- Every line in a proof is begins with a set of vertical bars.
- Relative to a certain line n, another line m is **accessible** if
  - m precedes n, and
  - all bars that include m also include n

good:



bad:



Natural deduction

- Rules: for every operator of statement logic, there are one or two **introduction** rules and one or two **elimination** rules
- Notation:
  - at least one formula or sub-proof above the horizontal line
  - one formula below the horizontal line
  - name of the rule is written next to the line

Natural deduction

- Rule application: if all formulas/sub-proofs over the line occur in a proof and are **accessible**, then the formula below the line may be added to the proof
- formulas in a proof are numbered
- the numbers of the used premises are written behind the new formula

Natural deduction: rules

Negation

$$\begin{array}{l|l} & \varphi \\ \hline & \vdots \\ & \psi \\ & \neg\psi \end{array}$$

$$\frac{\quad}{\neg\varphi}\ \neg I$$

$$\frac{\neg\neg\varphi}{\varphi}\ \neg E$$

Natural deduction: rules

Conjunction

$$\frac{\begin{array}{c}\varphi \\ \psi\end{array}}{\varphi \wedge \psi} \wedge I$$

$$\frac{\varphi \wedge \psi}{\varphi} \wedge E1 \qquad\qquad \frac{\varphi \wedge \psi}{\psi} \wedge E2$$

Natural deduction: rules

Disjunction

$$\frac{\varphi}{\varphi \vee \psi}\,\vee I1 \qquad\qquad \frac{\varphi}{\psi \vee \varphi}\,\vee I2$$

$$\frac{\begin{array}{c} \varphi \vee \psi \\ \begin{array}{|l} \varphi \\ \hline \vdots \\ \xi \end{array} \qquad \begin{array}{|l} \psi \\ \hline \vdots \\ \xi \end{array} \end{array}}{\xi}\,\vee E$$

Natural deduction: rules

Implication

$$\frac{\begin{array}{|l} \varphi \\ \hline \vdots \\ \psi \end{array}}{\varphi \rightarrow \psi} \rightarrow I$$

$$\frac{\begin{array}{c} \varphi \rightarrow \psi \\ \varphi \end{array}}{\psi} \rightarrow E$$

Natural deduction: rules

Equivalence

$$\dfrac{\begin{array}{|l} \varphi \\ \hline \vdots \\ \psi \end{array} \qquad \begin{array}{|l} \psi \\ \hline \vdots \\ \varphi \end{array}}{\varphi \leftrightarrow \psi} \leftrightarrow I$$

$$\dfrac{\begin{array}{c} \varphi \leftrightarrow \psi \\ \varphi \end{array}}{\psi} \leftrightarrow E, 1 \qquad \dfrac{\begin{array}{c} \varphi \leftrightarrow \psi \\ \psi \end{array}}{\varphi} \leftrightarrow E, 2$$

Natural deduction

Definition

If it is possible to construct a proof of the form

$$
\begin{array}{|l}
\varphi_1 \\
\vdots \\
\varphi_n \\
\hline
\vdots \\
\psi
\end{array}
$$

according to the rules of natural deduction, then ψ *is* **derivable** *from* $\varphi_1, \ldots, \varphi_n$, *i.e.*

$$\varphi_1, \ldots, \varphi_n \vdash \psi$$

Natural deduction

Theorem (Soundness and completeness)

$$M \vdash \varphi$$

if and only if

$$M \Rightarrow \varphi$$