Mathematics for linguists

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Natural deduction

- there is no simple algorithm to prove a given theorem/derivaiton
- you can always start a sub-proof with any arbitrary new hypotheses
- hence there are infinitely many proofs for each derivation
- **but:** it is not possible to prove via natural deduction that a formula is not derivable from a given set of premises
- if you suspect that the conclusion doesn't follow from the premises, it is safer to work with truth trees

Natural deduction

- always keep track which sub-goal you are currentyl proving
- if the current sub-goal is $\varphi \wedge \psi$:
  - first prove φ
  - then prove ψ
  - then apply $\wedge I$
- if the current sub-goal is $\neg\varphi$:
  - start a sub-proof with φ as additional assumption
  - for some convenient formula ψ: prove both ψ and $\neg\psi$
  - finish the sub-proof with $I\neg$

Natural deduction

- if the current sub-goal is $\varphi \rightarrow \psi$:
  - start a new sub-proof with φ as additional assumption
  - try to prove ψ
  - if successful: finish the sub-proof with $\rightarrow I$

Natural deduction

- if the current sub-goal is $\varphi \vee \psi$:
  - prove φ **or**
  - prove ψ
  - if successful, introduce $\varphi \vee \psi$ via $\vee I, 1(2)$

Natural deduction

- otherwise: if there is an accessible formula $\xi \vee \zeta$
  - combine $\vee E$ and $\vee I$:
  - start a sub-proof with the assumption ξ and prove φ (or ψ)
  - derive $\varphi \vee \psi$ using $\vee I$ and finish sub-proof
  - start a second sub-proof and prove ψ (φ)
  - from this, derive $\varphi \vee \psi$ via $\vee I$ and finish sub-proof
  - via $\vee E$, derive $\varphi \vee \psi$

Natural deduction

- if the currect sub-goal is $\varphi \leftrightarrow \psi$:
  - start sub-proof with the additional assumption φ
  - prove ψ
  - finish sub-proof and start new sub-proof withe the assumption ψ
  - prove φ
  - finish the second sub-proof and apply $\leftrightarrow I$

Natural deduction

- further rules of thumb:
  - apply $\wedge E$, $\rightarrow E$ and $\leftrightarrow E$ whenever possible
  - also, apply $\neg I$ as soon as possible; if the current line in the proof is the negation of an earlier accessible line, immediately end the current sub-proof with $\neg I$.

Natural deduction

- if none of these rules of thumb is applicable: **indirect proof**:
- suppose you want to prove φ
  - start your sub-proof with the assumption $\neg\varphi$
  - try to derive a contradiction
  - i.e.: try to derive both ψ and $\neg\psi$ for some formula ψ
  - if successful: end the current sub-proof with $\neg I$
  - result is $\neg\neg\varphi$
  - applying $\neg E$ leads to φ, as desired

Examples: de Morgan's Laws (1)

1	$\neg(p \wedge q)$	(A)
2	$\neg(\neg p \vee \neg q)$	(A)
3	$\neg p$	(A)
4	$\neg p \vee \neg q$	$\vee$ I 1;3
5	$\neg\neg p$	$\neg$ I; 3,4,2
6	$\neg q$	(A)
7	$\neg p \vee \neg q$	$\vee$ I 2;6
8	$\neg\neg q$	$\neg$ I; 6,7,2
9	p	$\neg$ E; 5
10	q	$\neg$ E; 8
11	$p \wedge q$	$\wedge$ I;9,10
12	$\neg\neg(\neg p \vee \neg q)$	$\neg$ I; 2,11,1
13	$\neg p \vee \neg q$	$\neg$ E; 12

$$\neg(p \wedge q) \vdash \neg p \vee \neg q$$

Examples: de Morgan's Laws (2)

1	$\neg p \vee \neg q$	(A)
2	$p \wedge q$	(A)
3	p	$\wedge$ I1;2
4	q	$\wedge$ I2;2
5	$\neg p$	(A)
6	$\neg p$	(6)
7	$\neg q$	(A)
8	p	(A)
9	p	8
10	$\neg p$	$\neg$ I;8,4,7
11	$\neg p$	$\vee$ E;1,5,6,7,9
12	$\neg(p \wedge q)$	$\neg$ I;2,3,11

$$\boxed{\neg p \vee \neg q \vdash \neg(p \wedge q)}$$

Examples: de Morgan's Laws (3)

1	$\neg(p \vee q)$	(A)
2	$\quad p$	(A)
3	$\quad p \vee q$	$\vee$ I1;2
4	$\neg p$	$\neg$ I;2,1,3
5	$\quad q$	(A)
6	$\quad p \vee q$	$\vee$ I2;5
7	$\neg q$	$\neg$ I;5,1,6
8	$\neg p \wedge \neg q$	$\wedge$ I; 4,7

$$\boxed{\neg(p \vee q) \vdash \neg p \wedge \neg q}$$

Examples: de Morgan's Laws (4)

1	$\neg p \wedge \neg q$	(A)
2	$\neg p$	$\wedge$ I1;1
3	$\neg q$	$\wedge$ I2;1
4	$p \vee q$	(A)
5	p	(A)
6	p	5
7	q	(A)
8	$\neg p$	(A)
9	$\neg p$	8
10	$\neg\neg p$	$\neg$ I;8,3,7
11	p	$\neg$ E; 10
12	p	$\vee$ E;4,5,6,7,11
13	$\neg(p \vee q)$	$\neg$ I; 4,2,12

$$\boxed{\neg p \wedge \neg q \vdash \neg(p \vee q)}$$

Summary: statement logic

- covered here: **classical statement logic**
- besides, there is a multitude of non-classical statement logics (intuitionistic logic, relevant logic, modal logics, linear logic, ...)

Summary: statement logic

- meta-logical properties of classical statement logic:
  - two-valued semantics (every statement is either true or false)
  - there is a sound and complete syntactic description of logical inference; there are several systems of syntactic rules (truth trees, natural deduction, ...) that identify exactly the set of tautologies
  - logical inference is **decidable**: there are mechanical decision procedures (for instance truth tables) that distinguish tautologies from non-tautologies

Summary: statement logic

- beyond statement logic:
  - classical first order logic (covered in the remainder of this course) has a sound and complete syntactic proof system, but is not decidable
  - second order logic (and higher order logics) and type theory are neither decidadble, nor do they have a complete syntactic proof system (i.e. it is not possible to describe the set of tautologies by means of finitely many syntactic rules)