Mathematics for linguists

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Quantifiers

- so far no significant extension of statement logic
- especially the theory of **logical inference** is identical to statement logic
- real quantum leap from statement logic to predicate logic is the introduction of **quantifiers**

Quantifiers

- PL (predicate logic) subsumes classical syllogistics

(1)
   a. All humans are mortal.
   b. No Greek is a philosopher.
   c. Some philosophers are musicians.
   d. Not all Greeks are musicians.

Expressions like *all, no, some, every, ...* are called **quantifiers**.

Quantoren

- PL extends syllogistics in two ways:
  - several quantifiers can occur within one simple statement

(2) Every Greek knows some musician.

  - bound pronouns/variables

(3) For *every Greek* it holds that: if *he* knows some musician, then *he* knows some instrument.

The universal quantifier

- new symbol: $\forall$
- pronounced as: "for all" or "for every"
- direct counterpart of English *for every object, it holds that:*
- Engl.: *every object* is referred to via pronoun *it*
- PL:
  - pronouns are translated as variables
  - for clarity's sake, it is indicated at the quantifier which variable it binds

The universal quantifier

For every object it holds: if it is a triangle, it is a polygon.

$$\rightsquigarrow$$

$$\forall x(\mathit{Triangle}(x) \to \mathit{Polygon}(x))$$

For each object it holds: it is a Greek, or it is not a Greek.

$$\rightsquigarrow$$

$$\forall y(\mathit{Greek}(y) \vee \neg\mathit{Greek}(y))$$

The universal quantifier

By means of appropriate paraphrases, expressions like *all* and *every* can be translated using the universal quantifier. For instance:

- original sentence

  All humans are mortal.

- paraphrase:

  For each object it holds: if it is human, it is mortal.

- translation:

$$\forall x(\mathit{Human}(x) \to \mathit{Mortal}(x))$$

The existential quantifier

- new symbol: $\exists$
- pronounced as: "there is a" or "there exists a"
- PL-counterpart to English *There is an object such that*
- as with the universal quantifier, it is indicated explicitly which variable is bound

The existential quantifier

There is an object such that it is a rectangel and a rhombus. $\leadsto$
$\exists x(\mathit{Rectangle}(x) \wedge \mathit{Rhombus}(x))$

There is an object such that it is a Greek but not a philosopher. $\leadsto$
$\exists z(\mathit{Greek}(z) \wedge \neg \mathit{Philosopher}(z))$

The existential quantifier

By means of appropriate paraphrases, expressions like *some* and *a* can be translated using the existential quantifier. For instance:

- original sentence:

  Some Greeks are philosophers.

- paraphrase:

  There is an object such that it is a Greek and a philosopher.

- translation:

  $$\exists y(\mathit{Greek}(y) \wedge \mathit{Philosopher}(y))$$

Restricted quantification

- Quantification in natural language is usually **restricted**

  All **Humans** *are mortal.*

  Some **Greeks** *are philosophers.*

- quantification in logic is in principle **unrestricted**

  for every **object**, *there is an* **object**

- Restriction of the universal quantifier is translated using the **implication**

$$\forall x(Human(x) \rightarrow Mortal(x))$$

- Restriction of the existential quantifier is translated using **conjunction**

$$\exists x(Greek(x) \wedge Philosopher(x))$$

Multiple quantification

- One sentence may contain more than one quantifying expression

(4) a. Every man loves every dish.
b. All children read all books.
c. Some children gave a guest a candy.

- Accordingly, translation contains several quantifiers.

(5) a. $\forall x(\mathit{Man}(x) \to \forall y(\mathit{Dish}(y) \to \mathit{Love}(x, y)))$
b. $\forall x(\mathit{Child}(x) \to \forall y(\mathit{Book}(y) \to \mathit{Read}(x, y)))$
c. $\exists x(\mathit{Child}(x) \wedge \exists y(\mathit{Guest}(y) \wedge \exists z(\mathit{Candy}(z) \wedge \mathit{Give}(x, y, z))))$

Rules of thumb for translation

- given: English sentence S that needs a quantifier to be translated
- paraphrase S in such a way that it starts with *for all P it holds that ...* or *there is a P such that ...* (where "P" is a noun)
- translate as

  $$\forall x(P(x) \to ...)$$

  or

  $$\exists x(P(x) \wedge ...)$$

  ("P" is the translation of the noun in question
- translate the rest of the sentence

Example

(6) a. Dogs are intelligent.

(7) a. Every man cheats himself.

(8) a. Lions have a mane.

Example

(6) a. Dogs are intelligent.
b. For every dog it holds that it is intelligent.

(7) a. Every man cheats himself.
b. For every man it holds that he cheats himself.

(8) a. Lions have a mane.
b. For every lion it holds that there is a mane such that it has it.

Example

(6) a. Dogs are intelligent.
b. For every dog it holds that it is intelligent.
c. $\forall x(\mathit{Dog}(x) \rightarrow \mathit{Intelligent}(x))$

(7) a. Every man cheats himself.
b. For every man it holds that he cheats himself.
c. $\forall x(\mathit{Man}(x) \rightarrow \mathit{Cheat}(x, x)x)$

(8) a. Lions have a mane.
b. For every lion it holds that there is a mane such that it has it.
c. $\forall y(\mathit{Lion}(y) \rightarrow \exists w(\mathit{Mane}(w) \wedge \mathit{Has}(y, w)))$

Scope ambiguity

- Sentences with more than one quantifier can be **ambiguous**
- Expressions of predicate logic are never ambiguous
- ambiguous sentences thus have more than one translation

Every man loves a woman.

$\forall x(\mathit{Man}(x) \rightarrow \exists y(\mathit{Woman}(y) \wedge \mathit{Loves}(x,y)))$ $\quad$ $\exists y(\mathit{Woman}(y) \wedge \forall x(\mathit{Man}(y) \rightarrow \mathit{Loves}(x,y)))$

Syntax of predicate logic

Definition (Syntax of predicate logic, final version)

1. *There are infinitely many individual constants.*
2. *There are infinitely many individual variables.*
3. *Every individual constant and every individual variable is a term.*
4. *For every natural number n there are infinitely many n-place predicates.*
5. *If P is an n-place predicate and $t_1, \ldots, t_n$ are terms, then $P(t_1, \ldots, t_n)$ is an atomic formula.*
6. *If t_1 and t_2 are terms, $t_1 = t_2$ is an atomic formula.*
7. *Every atomic formula is a formula.*
8. *If φ and ψ are formulas, then $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \rightarrow \psi$ and $\varphi \leftrightarrow \psi$ are also formulas.*
9. *If v is a variable and φ a formula, then $\forall v(\varphi)$ and $\exists v(\varphi)$ are also formulas.*

Syntax of PL: conventions

- The bracketing conventions of statement logic hold.
- Furthermore, it holds that $\forall v$ and $\exists v$ associate stronger than all other operators.

$$\forall x Px \wedge Qx$$

abbreviates

$$\forall x(P(x)) \wedge Q(x),$$

not

$$\forall x(P(x) \wedge Q(x))!$$

Free and bound variables

- we distinguish **free** and bound occurrences of variables in a formula
- bound occurrences of a variable in a formula are always bound **by a particular quantifier**

Free and bound variables

Definition (Free and bound variable occurrences)

- *All variable occurrence in an atomic formula* φ *are free in* φ.
- *Every free occurrence of a variable in* v *in* φ *is also freee in* $\neg\varphi$.
- *Every free occurrence of a variable* v *in* φ *and* ψ *is also free in in* $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \rightarrow \psi$ *and* $\varphi \leftrightarrow \psi$.
- *Every free occurrence of a variable* v *in* φ *is also free in* $\forall w(\varphi)$ *and* $\exists w(\varphi)$*, if* $v \neq w$.
- *Every free occurrence of a variable* v *in* φ *is*
    - *bound in* $\forall v(\varphi)$ *by* $\forall v$*, and*
    - *bound in* $\exists v(\varphi)$ *by* $\exists v$.
- *If a variable occurrence* v *is bound in* φ*, it is also bound in every formula that contains* φ *as a sub-formula.*

Bound variables and scope

- The formula within the bracket pair after a quantifier is called the **scope of the quantifier**
- Example (quantifier in blue, scope in red)

$$\forall x(P(x) \to Q(x))$$
$$\forall x(P(x) \to Q(x)) \wedge Q(x)$$
$$\exists x(R(x)) \wedge \forall x(P(x) \to Q(x))$$
$$\exists x(R(x) \wedge \forall x(P(x) \to Q(x)))$$

- A quantifier Q binds a variable occurrence v iff
  - v occurs in the scope of Q, and
  - between Q and v there is no intervening **co-indexed** quantifier Q' such that v is in the scope of Q' (and that would therefore bind v)

Predicate logic: another example

$$
\begin{aligned}
M &= \langle E, F\rangle \\
E &= \{\textbf{DOG}, \textbf{CAT}, \textbf{MAN}_1, \textbf{MAN}_2, \textbf{WOMAN}_1, \\
&\quad \textbf{WOMAN}_2, \textbf{CAKE}, \textbf{MOUSE}\} \\
F(\mathit{jo}) &= \textbf{MAN}_1 \\
F(\mathit{bertie}) &= \textbf{MAN}_2 \\
F(\mathit{ethel}) &= \textbf{WOMAN}_1 \\
F(\mathit{fiona}) &= \textbf{WOMAN}_2 \\
F(\mathit{chester}) &= \textbf{DOG} \\
F(\mathit{prudence}) &= \textbf{CAT}
\end{aligned}
$$

Predicate logic: another example

$$\begin{aligned}
F(\mathit{Animal}) &= \{\textbf{DOG, CAT, MOUSE}\}\\
F(\mathit{Run}) &= \{\textbf{DOG, CAT}\}\\
F(\mathit{Laugh}) &= \{\textbf{MAN}_1, \textbf{WOMAN}_1\}\\
F(\mathit{Howl}) &= \{\textbf{DOG}\}\\
F(\mathit{Sing}) &= \{\textbf{WOMAN}_2\}\\
F(\mathit{Scream}) &= \emptyset\\
F(\mathit{Squeak}) &= \{\textbf{MOUSE}\}\\
F(\mathit{Crazy}) &= \emptyset\\
F(\mathit{Poison}) &= \{\langle\textbf{CAKE, DOG}\rangle\}\\
F(\mathit{Eat}) &= \{\langle\textbf{DOG, CAKE}\rangle\}
\end{aligned}$$

Universal quantifier: interpretation

- **notational convention:**

  $$[t/v]\varphi$$

  is the formula that is exactly like φ except that all **free** occurrences of the variable v are replaced by t

Universal quantifier: interpretation

- Intuition:

$$\forall v \varphi$$

is true if and only if $[c/v]\varphi$ is true for all individual constants c

- But: in our model

$$\textit{Animal}(c) \to \textit{Run}(c)$$

holds for all individual constants c; still

$$\forall x(\textit{Animal}(x) \to \textit{Run}(x))$$

is false!

- Reason: the mouse "has no name"

Universal quantifier: interpretation

- second attempt: to make

$$\forall x(\mathit{Animal}(x) \to \mathit{Run}(x))$$

  true,

$$\mathit{Animal}(x) \to \mathit{Run}(x)$$

  must be true, no matter what x refers to!
- Suppose, $g(x) = \mathbf{MOUSE}$
- then:

$$[\mathit{Animal}(x) \to \mathit{Run}(x)]_g^M = 0$$

Universal quantifier: interpretation

- perhaps:

$$[\forall v(\varphi)]^M = 1$$

  if and only if for all g:

$$[\varphi]^M_g = 1$$

- But what about formulas like

$$\forall x \neg \forall y \mathit{Poison}(x, y)$$

Universal quantifier: interpretation

- two problems:
  - quantified formulas may contain free variables; therefore their interpretation must depend on the assignment function as well
  - not the entire assignment function is varied by a quantifier, but only the interpretation of the bound variable

Universal quantifier: interpretation

- Notation:
  - let $a \in E$ be an object of the model, v a variable and g an assignment function
  - $g[a/v]$: the assignment function that is exactly like g except that

$$g[a/v](v) = a$$

- final version: Let $M = \langle E, F \rangle$ be a model.

$$[\forall v(\varphi)]_g^M = 1$$

if and only if

$$[\varphi]_{g[a/v]}^M = 1$$

for all $a \in E$

Existential quantifier: interpretation

- Intuition:

$$\exists v(\varphi)$$

is true if and only if there is some individual constant c such that

$$[c/v]\varphi$$

is true

- but:

$$\exists x(\mathit{Squeak}(x))$$

is (intuitively) true in our model even though there is no individual constant c in our example such that

$$\mathit{Squeak}(c)$$

would be true in the model.

Existential quantifier: interpretation

- problem can be avoided via varying the assignment function as well:

$$[\exists v(\varphi)]_g^M = 1$$

if and only if there is an object $a \in E$ such that

$$[\varphi]_{g[a/v]}^M = 1$$

- in the example we have

$$[\mathit{Squeak}(x)]_{g[\mathbf{MOUSE}/x]}^M = 1$$

and hence the quantified formula is true.

Semantics of predicate logic

Definition (Semantics of predicate logic (final version))

Let $M = \langle E, F\rangle$ be a model and g an assignment function for M.

1. $[c]^M_g = F(c)$, *if c is an individual constant.*
2. $[v]^M_g = g(v)$, *if v is an individual variable.*
3. $[P(t_1, \ldots, t_n)]^M_g = 1$ *iff* $\langle [t_1]^M_g, \ldots, [t_n]^M_g\rangle \in F(P)$
4. $[t_1 = t_2]^M_g$ *iff* $[t_1]^M_g = [t_2]^M_g$
5. $[\neg\varphi]^M_g = 1 - [\varphi]^M_g$
6. $[\varphi \wedge \psi]^M_g = \min([\varphi]^M_g, [\psi]^M_g)$
7. $[\varphi \vee \psi]^M_g = \max([\varphi]^M_g, [\psi]^M_g)$
8. $[\varphi \rightarrow \psi]^M_g = \max(1 - [\varphi]^M_g, [\psi]^M_g)$
9. $[\varphi \leftrightarrow \psi]^M_g = 1 - ([\varphi]^M_g - [\psi]^M_g)^2$
10. $[\forall v(\varphi)]^M_g = \min(\{[\varphi]^M_{g[a/v]} | a \in E\})$
11. $[\exists v(\varphi)]^M_g = \max(\{[\varphi]^M_{g[a/v]} | a \in E\})$