Mathematics for linguists

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Natural deduction for predicate logic

- direct extension of natural deduction for statement logic
- four new rules: one introduction rule and one elimination rule for each quantifier
- there are side conditions that need to be taken into account

Natural deduction: rules

Universal quantifier

$$\frac{\varphi}{\forall v \varphi} \forall I$$

- v is an arbitrary variable
- **Constraint:** v does not occur free in any accessible assumption!

$$\frac{\forall v \varphi}{[t/v]\varphi} \forall E$$

- v is an arbitrary variable and t an arbitrary constant or variable
- **Constraint:** if t is a variable, it must not occur bound in $[t/v]\varphi$

Natural deduction: rules

Existential quantifier

$$\frac{[t/v]\varphi}{\exists v\varphi}\ \exists I$$

- v is an arbitrary variable and t an arbitrary constant or variable
- **Constraint:** if t is a variable, it must not occur bound in $[t/v]\varphi$

Natural deduction: rules

Existential quantifier

$$
\begin{array}{l}
\exists v \varphi \\
\left|\begin{array}{l}
[c/v]\varphi \\ \hline
\vdots \\
\psi
\end{array}\right. \\ \hline
\quad \psi
\end{array} \; \exists E
$$

- v is an arbitrary variable
- constraints
  - c is a new constant that does not occur so far in the proof
  - c does not occur in ψ

Examples

$$\neg\exists x Px \vdash \forall x \neg Px$$

1	$\neg\exists x P(x)$	(A)
2	$\quad Px$	(A)
3	$\quad \exists x Px$	2;$\exists I$
4	$\neg Px$	2,3,1,3; $\neg I$
5	$\forall x \neg Px$	4; $\forall I$

Examples

$$\forall x \neg Px \vdash \neg \exists x Px$$

1	$\forall x \neg Px$	(A)
2	$\quad \exists x Px$	(A)
3	$\quad\quad Pa$	
4	$\quad\quad\quad \forall x \neg Px$	(A)
5	$\quad\quad\quad \neg Pa$	4;$\forall E$
6	$\quad\quad \neg\forall x \neg Px$	4,5,3,5; $\neg I$
7	$\quad \neg\forall x \neg Px$	2,3,4;
8	$\neg \exists x Px$	2,1,7;$\neg I$

Examples

$$\neg\forall x Px \vdash \exists x \neg Px$$

1	$\neg\forall x Px$	(A)
2	$\quad \neg\exists x\neg Px$	(A)
3	$\quad\quad \neg Px$	(A)
4	$\quad\quad \exists x\neg Px$	3; $\exists I$
5	$\quad \neg\neg Px$	3,4,2;$\neg I$
6	$\quad Px$	$\neg E$
7	$\quad \forall x Px$	6;$\forall I$
8	$\neg\neg\exists x\neg Px$	2,7,1; $\neg I$
9	$\exists x\neg Px$	$\neg E$

Examples

$$\exists x \neg Px \vdash \neg \forall x Px$$

1	$\exists x \neg Px$	(A)
2	$\quad \forall x Px$	(A)
3	$\quad\quad \neg Pa$	(A)
4	$\quad\quad\quad \exists x \neg Px$	(A)
5	$\quad\quad\quad Pa$	2;$\forall E$
6	$\quad\quad \neg \exists x \neg Px$	4,3,5;$\neg I$
7	$\quad \neg \exists x \neg Px$	1,2,3;$\exists E$
8	$\neg \forall x Px$	2,3,1;$\neg I$

Examples

$$\forall x(Px \wedge Qx) \vdash \forall xPx \wedge \forall xQx$$

1	$\forall x(Px \wedge Qx)$	(A)
2	$Px \wedge Qx$	1; $\forall E$
3	Px	2;$\wedge E1$
4	Qx	2;$\wedge E2$
5	$\forall xPx$	3; $\forall I$
6	$\forall xQx$	4;$\forall I$
7	$\forall xPx \wedge \forall xQx$	5,4;$\wedge I$

Examples

$$\exists x Px \rightarrow Qa \vdash \forall x(Px \rightarrow Qa)$$

1	$\exists x Px \rightarrow Qa$	(A)
2	$\quad Px$	(A)
3	$\quad \exists x Px$	2;$\exists I$
4	$\quad Qa$	1,2;$\rightarrow E$
5	$Px \rightarrow Qa$	2,3; $\rightarrow I$
6	$\forall x(Px \rightarrow Qa)$	5; $\forall I$

Final remarks

- calculus of natural deduction is sound and complete
- this means that all and only the logically valid inferences can be proved
- the constraints are necessary; otherwise it would be possible to derive invalid inferences, for instance
    - $\exists xPx \vdash \forall xPx$

Final remarks

- As for the truth tree method, there is no fool-proof solution strategy for natural deduction; and for the same reason
- with the elimination rule for the existential quantifier, arbitrarily many constants can be introduced into a proof, and each constant can be used in the elimination rule for the universal quantifier