Mathematics for linguists

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Natural deduction for predicate logic

- direct extension of natural deduction for statement logic
- four new rules: one introduction rule and one elimination rule for each quantifier
- there are side conditions that need to be taken into account

Natural deduction: rules

Universal quantifier

$$\frac{\varphi}{\forall v\varphi}\,\forall I$$

- ullet v is an arbitrary variable
- Constraint: v does not occur free in any accessible assumption!

$$\frac{\forall v\varphi}{[t/v]\varphi}\,\forall E$$

- ullet v is an arbitrary variable and t an arbitrary constant or variable
- \bullet Constraint: if t is a variable, it must not occur bound in $[t/v]\varphi$

Natural deduction: rules

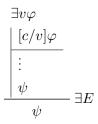
Existential quantifier

$$\frac{[t/v]\varphi}{\exists v\varphi}\,\exists I$$

- ullet v is an arbitrary variable and t an arbitrary constant or variable
- ullet Constraint: if t is a variable, it must not occur bound in [t/v] arphi

Natural deduction: rules

Existential quantifier



- ullet v is an arbitrary variable
- constraints
 - ullet c is a new constant that does not occur so far in the proof
 - ullet c does not occur in ψ

$$\forall x \neg Px \vdash \neg \exists x Px$$

$$1 \qquad \forall x \neg Px \qquad (A)$$

$$2 \qquad \exists x Px \qquad (A)$$

$$3 \qquad Pa$$

$$4 \qquad \forall x \neg Px \qquad (A)$$

$$5 \qquad \neg Pa \qquad 4; \forall E$$

$$6 \qquad \neg \forall x \neg Px \qquad 4,5,3,5; \neg I$$

$$7 \qquad \neg \forall x \neg Px \qquad 2,3,4;$$

$$8 \qquad \neg \exists x Px \qquad 2,1,7; \neg I$$

$$\exists x \neg Px \vdash \neg \forall x Px$$

$$1 \qquad \exists x \neg Px \qquad (A)$$

$$2 \qquad \forall x Px \qquad (A)$$

$$3 \qquad \neg Pa \qquad (A)$$

$$4 \qquad \exists x \neg Px \qquad (A)$$

$$5 \qquad Pa \qquad 2; \forall E$$

$$6 \qquad \neg \exists x \neg Px \qquad 4,3,5; \neg I$$

$$7 \qquad \neg \exists x \neg Px \qquad 1,2,3; \exists E$$

$$8 \qquad \neg \forall x Px \qquad 2,3,1; \neg I$$

$$\exists x Px \to Qa \vdash \forall x (Px \to Qa)$$

$$1 \qquad \exists x Px \to Qa \qquad (A)$$

$$2 \qquad Px \qquad (A)$$

$$3 \qquad \exists x Px \qquad 2; \exists I$$

$$4 \qquad Qa \qquad 1, 2; \to E$$

$$5 \qquad Px \to Qa \qquad 2, 3; \to I$$

$$6 \qquad \forall x (Px \to Qa) \qquad 5; \forall I$$

Final remarks

- calculus of natural deduction is sound and complete
- this means that all and only the logically valid inferences can be proved
- the constraints are necessary; otherwise it would be possible to derive invalid inferences, for instance
 - \bullet $\exists x P x \vdash \forall x P x$

Final remarks

- As for the truth tree method, there is no fool-proof solution strategy for natural deduction; and for the same reason
- with the elimination rule for the existential quantifier, arbitrarily many constants can be introduced into a proof, and each constant can be used in the elimination rule for the universal quantifier