Mathematics for linguists

Gerhard Jäger

University of Tübingen

October 26, 2010

EBERHARD KARLS
UNIVERSITÄT
TÜBINGEN

Review: The Syntax of Statement Logic

We have $p, q, r, ...$ as atomic statements.

Review: The Syntax of Statement Logic

We have $p, q, r, ...$ as atomic statements.
The syntax of statement logic is defined *inductively*:

Review: The Syntax of Statement Logic

We have $p, q, r, ...$ as atomic statements.
The syntax of statement logic is defined *inductively*:

- Every atomic statement p is a statement.

Review: The Syntax of Statement Logic

We have $p, q, r, ...$ as atomic statements.
The syntax of statement logic is defined *inductively*:

- Every atomic statement p is a statement.
- If φ is a statement, $\neg\varphi$ also is a statement.

Review: The Syntax of Statement Logic

We have $p, q, r, ...$ as atomic statements.
The syntax of statement logic is defined *inductively*:

- Every atomic statement p is a statement.
- If φ is a statement, $\neg\varphi$ also is a statement.
- If φ and ψ are statements, $(\varphi \wedge \psi)$ also is a statement.

Review: The Syntax of Statement Logic

We have $p, q, r, ...$ as atomic statements.
The syntax of statement logic is defined *inductively*:

- Every atomic statement p is a statement.
- If φ is a statement, $\neg\varphi$ also is a statement.
- If φ and ψ are statements, $(\varphi \wedge \psi)$ also is a statement.

Nothing else is a statement.

Review: The Syntax of Statement Logic

We have $p, q, r, ...$ as atomic statements.
The syntax of statement logic is defined *inductively*:

- Every atomic statement p is a statement.
- If φ is a statement, $\neg\varphi$ also is a statement.
- If φ and ψ are statements, $(\varphi \wedge \psi)$ also is a statement.

Nothing else is a statement.
Note that we assume that statements are finite.

Natural Numbers

Note that the natural numbers are also defined inductively:

Natural Numbers

Note that the natural numbers are also defined inductively:

- 0 is a natural number.

Natural Numbers

Note that the natural numbers are also defined inductively:

- 0 is a natural number.
- If n is a natural number, $succ(n)[= n + 1]$ also is a natural number.

Natural Numbers

Note that the natural numbers are also defined inductively:

- 0 is a natural number.
- If n is a natural number, $succ(n)[= n + 1]$ also is a natural number.

Nothing else is a natural number.

Natural Numbers

Note that the natural numbers are also defined inductively:

- 0 is a natural number.
- If n is a natural number, $succ(n)[= n + 1]$ also is a natural number.

Nothing else is a natural number.
Again: Finiteness.

Natural Numbers

Note that the natural numbers are also defined inductively:

- 0 is a natural number.
- If n is a natural number, $succ(n)[= n + 1]$ also is a natural number.

Nothing else is a natural number.
Again: Finiteness.
Generally, proving by induction that "All X have the property P" is possible just in case the set of all X is defined inductively.

Note

Note that an inductive definition always consists of

- One or more *basic cases*.

Note

Note that an inductive definition always consists of

- One or more *basic cases*.
- One or more rules on how to construct new objects from objects that have already been constructed.

Proof by Induction on the Structure of Statements

We want to show that a property P holds for all statements.

Proof by Induction on the Structure of Statements

We want to show that a property P holds for all statements.
Idea: We

- Show that P holds for the most basic (atomic) statements.

Proof by Induction on the Structure of Statements

We want to show that a property P holds for all statements.
Idea: We

- Show that P holds for the most basic (atomic) statements.
- Show that every possibility of building complex statements from statements that have already been constructed 'transfers' P from the constituent statement to the complex one.

Example

Theorem: In every statement φ of statement logic, the number of occurences of binary connectives equals the number of occurences of atomic statements -1.

Example

Theorem: In every statement φ of statement logic, the number of occurences of binary connectives equals the number of occurences of atomic statements -1.
Proof:

Base: Let $\varphi \equiv p$;

Example

Theorem: In every statement φ of statement logic, the number of occurences of binary connectives equals the number of occurences of atomic statements -1.

Proof:

Base: Let $\varphi \equiv p$; the property holds.

Example

Theorem: In every statement φ of statement logic, the number of occurences of binary connectives equals the number of occurences of atomic statements -1.

Proof:

Base: Let $\varphi \equiv p$; the property holds.

Step:
- Suppose the property holds for ψ and let $\varphi \equiv \neg\psi$. Then it also holds for φ, since prefixing "$\neg$" to ψ neither changes the number of occurences of atomic statements nor of binary connectives.

Example

Theorem: In every statement φ of statement logic, the number of occurences of binary connectives equals the number of occurences of atomic statements -1.

Proof:

Base: Let $\varphi \equiv p$; the property holds.

Step:
- Suppose the property holds for ψ and let $\varphi \equiv \neg\psi$. Then it also holds for φ, since prefixing "$\neg$" to ψ neither changes the number of occurences of atomic statements nor of binary connectives.
- Suppose the property holds for ψ and θ and let $\varphi \equiv (\psi \wedge \theta)$

Example

Theorem: In every statement φ of statement logic, the number of occurences of binary connectives equals the number of occurences of atomic statements -1.

Proof:

Base: Let $\varphi \equiv p$; the property holds.

Step:
- Suppose the property holds for ψ and let $\varphi \equiv \neg\psi$. Then it also holds for φ, since prefixing "$\neg$" to ψ neither changes the number of occurences of atomic statements nor of binary connectives.
- Suppose the property holds for ψ and θ and let $\varphi \equiv (\psi \wedge \theta)$Let the number of atomic statements in ψ be m, that of atomic statements in θ be n.

Example

Theorem: In every statement φ of statement logic, the number of occurences of binary connectives equals the number of occurences of atomic statements -1.

Proof:

Base: Let $\varphi \equiv p$; the property holds.

Step:
- Suppose the property holds for ψ and let $\varphi \equiv \neg\psi$. Then it also holds for φ, since prefixing "$\neg$" to ψ neither changes the number of occurences of atomic statements nor of binary connectives.
- Suppose the property holds for ψ and θ and let $\varphi \equiv (\psi \wedge \theta)$Let the number of atomic statements in ψ be m, that of atomic statements in θ be n.Then in φ there are $m + n$ occurrences of atomic statements.

Example

Theorem: In every statement φ of statement logic, the number of occurences of binary connectives equals the number of occurences of atomic statements -1.

Proof:

Base: Let $\varphi \equiv p$; the property holds.

Step:
- Suppose the property holds for ψ and let $\varphi \equiv \neg\psi$. Then it also holds for φ, since prefixing "$\neg$" to ψ neither changes the number of occurences of atomic statements nor of binary connectives.
- Suppose the property holds for ψ and θ and let $\varphi \equiv (\psi \wedge \theta)$Let the number of atomic statements in ψ be m, that of atomic statements in θ be n.Then in φ there are $m + n$ occurrences of atomic statements.By the induction assumption, $m - 1$ binary connectives occur in ψ and $n - 1$ occur in θ.

Example

Theorem: In every statement φ of statement logic, the number of occurences of binary connectives equals the number of occurences of atomic statements -1.

Proof:

Base: Let $\varphi \equiv p$; the property holds.

Step:
- Suppose the property holds for ψ and let $\varphi \equiv \neg\psi$. Then it also holds for φ, since prefixing "$\neg$" to ψ neither changes the number of occurences of atomic statements nor of binary connectives.
- Suppose the property holds for ψ and θ and let $\varphi \equiv (\psi \wedge \theta)$Let the number of atomic statements in ψ be m, that of atomic statements in θ be n.Then in φ there are $m + n$ occurrences of atomic statements.By the induction assumption, $m - 1$ binary connectives occur in ψ and $n - 1$ occur in θ.Since one "$\wedge$" is added in φ that neither occurs in ψ nor in θ, the total number of occurrences of binary connectives in φ is $(m - 1) + (n - 1) + 1 = m + n - 1$.

Example

Theorem: In every statement φ of statement logic, the number of occurences of binary connectives equals the number of occurences of atomic statements -1.

Proof:

Base: Let $\varphi \equiv p$; the property holds.

Step:
- Suppose the property holds for ψ and let $\varphi \equiv \neg\psi$. Then it also holds for φ, since prefixing "$\neg$" to ψ neither changes the number of occurences of atomic statements nor of binary connectives.
- Suppose the property holds for ψ and θ and let $\varphi \equiv (\psi \wedge \theta)$Let the number of atomic statements in ψ be m, that of atomic statements in θ be n.Then in φ there are $m + n$ occurrences of atomic statements.By the induction assumption, $m - 1$ binary connectives occur in ψ and $n - 1$ occur in θ.Since one "$\wedge$" is added in φ that neither occurs in ψ nor in θ, the total number of occurrences of binary connectives in φ is $(m - 1) + (n - 1) + 1 = m + n - 1$.So the property holds for φ.

Example

Theorem: In every statement φ of statement logic, the number of occurences of binary connectives equals the number of occurences of atomic statements -1.

Proof:

Base: Let $\varphi \equiv p$; the property holds.

Step:
- Suppose the property holds for ψ and let $\varphi \equiv \neg\psi$. Then it also holds for φ, since prefixing "$\neg$" to ψ neither changes the number of occurences of atomic statements nor of binary connectives.
- Suppose the property holds for ψ and θ and let $\varphi \equiv (\psi \wedge \theta)$Let the number of atomic statements in ψ be m, that of atomic statements in θ be n.Then in φ there are $m + n$ occurrences of atomic statements.By the induction assumption, $m - 1$ binary connectives occur in ψ and $n - 1$ occur in θ.Since one "$\wedge$" is added in φ that neither occurs in ψ nor in θ, the total number of occurrences of binary connectives in φ is $(m - 1) + (n - 1) + 1 = m + n - 1$.So the property holds for φ.

Since we have considered all cases, we have established the theorem.