Mathematics for linguists

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Composition of relations and functions

- let $R \subseteq A \times B$ and $S \subseteq B \times C$ be relations
- new relation $S \circ R \subseteq A \times C$ is formed as

$$S \circ R := \{\langle x, y \rangle | \text{there is a } z \text{ such that } R(x, z) \text{ and } S(z, y)\}$$

- If R and S are functions, $S \circ R$ is also a function.

The identity function

- function $F : A \to A$ is called **identity function** iff

$$F = \{\langle x, x\rangle \in A \times A | x \in A\}$$

- notation: $F = id_A$
- composition of a relation R with an identity function with the appropriate domain yields R again:
  - If $R \subseteq A \times B$, then
    - $id_B \circ R = R$
    - $R \circ id_A = R$
- If $F : A \to B$ is an injective function, then:
  - $F \circ F^{-1} \subseteq id_B$
  - $F^{-1} \circ F = id_A$
- Generally, it holds that:
  - $id_{\pi_0[R]} \subseteq R^{-1} \circ R$
  - $id_{\pi_1[R]} \subseteq R \circ R^{-1}$

Properties of relations

Many relations that are used in linguistic applications have certain structural properties. The most important ones are briefly discussed here; the apply to relations on a given set, i.e. $R \subseteq A \times A$.

Reflexivity

$R \subseteq A \times A$ is **reflexive** iff for all $x \in A$ it holds that $R(x, x)$.

Reflexivity

Example

- $A = \{1, 2, 3\}$
- $R_1 = \{\langle 1, 1\rangle, \langle 2, 2\rangle, \langle 3, 3\rangle, \langle 3, 1\rangle\}$
- $R_2 = \{\langle 1, 1\rangle, \langle 2, 2\rangle\}$

R_1 is reflexive but R_2 is not (because $\langle 3, 3\rangle \notin R_2$).

$R \subseteq A \times A$ is reflexive iff $id_A \subseteq R$.

More examples

- "has the same final digit as"
- "has birthday on the same day as"
- "greater or equal"

Irreflexivity

- Relations that are not reflexive are called **non-reflexive**
- Relations that **never** connect an object to itself are called **irreflexive**.

Irreflexivity

R is irreflexive iff there is no object x such that $R(x, x)$.

In other words: R is irreflexive iff $R \cap id_A = \emptyset$.

Examples

- $R_3 = \{\langle 1, 2\rangle, \langle 3, 2\rangle\}$
- $R_4 = \{\langle a, b\rangle \in \mathbb{N} \times \mathbb{N} | a < b\}$

Symmetry

A relation R is **symmetric** iff for all x, y with $R(x, y)$ it holds that $R(y, x)$.

In other words: R is symmetric iff $R = R^{-1}$.

Examples

- "married to"
- "relatively prime"
- "happened in the same year as"
- "cousin of"
- $\{\langle 1, 2\rangle, \langle 2, 1\rangle, \langle 3, 2\rangle, \langle 2, 3\rangle\}$
- $\{\langle 1, 3\rangle, \langle 3, 1\rangle\}$
- $\{\langle 2, 2\rangle\}$

Asymmetry

Asymmetry

A relation R is **asymmetric** iff it never holds that both $R(x, y)$ and $R(y, x)$.

- Every asymmetric relation must be irreflexive.
- examples:
  - $\{\langle 2, 3\rangle, \langle 1, 2\rangle\}$
  - $\{\langle 1, 3\rangle, \langle 2, 3\rangle, \langle 1, 2\rangle\}$
  - $\{\langle 1, 2\rangle\}$

Anti-symmetry

Anti-symmetry

A relation R is **anti-symmetric** iff whenever both $R(x, y)$ and $R(y, x)$, then $x = y$.

- R need not be reflexive to be anti-symmetric!
- Every asymmetric relation is also anti-symmetric.
- If R is anti-symmetric, then $R - id_A$ is asymmetric.

Anti-symmetry

Examples for anti-symmetric relations

- "greater or equal"
- "divisible by"
- "is a subset of"
- $\{\langle 2, 3\rangle, \langle 1, 1\rangle\}$
- $\{\langle 1, 1\rangle, \langle 2, 2\rangle\}$
- $\{\langle 1, 2\rangle, \langle 2, 3\rangle\}$

Transitivity

Transitivity

A relation R is **transitive** iff whenever $R(x, y)$ and $R(y, z)$, then $R(x, z)$.

- examples:
  - "older than"
  - "richer than"
  - "greater than"
  - "ancestor of"
  - equality
  - $\{\langle 2, 2\rangle\}$
  - $\{\langle 1, 2\rangle, \langle 2, 3\rangle, \langle 1, 3\rangle\}$
  - $\{\langle 1, 2\rangle, \langle 2, 1\rangle, \langle 1, 1\rangle, \langle 2, 2\rangle\}$
  - $\{\langle 1, 2\rangle, \langle 2, 3\rangle, \langle 1, 3\rangle, \langle 3, 2\rangle, \langle 2, 1\rangle, \langle 3, 1\rangle, \langle 1, 1\rangle, \langle 2, 2\rangle, \langle 3, 3\rangle\}$
- R is transitive iff $R \circ R \subseteq R$.

Connectedness

Connectedness

A relation $R \subseteq A \times A$ is **connected** iff for all $x, y \in A$ with $x \neq y$: $R(x, y)$ or $R(y, x)$ (or both).

Examples

- "greater than" (as applied to the natural numbers)
- $\{\langle 1, 2\rangle, \langle 3, 1\rangle, \langle 3, 2\rangle\}$
- $\{\langle 1, 1\rangle, \langle 2, 3\rangle, \langle 1, 2\rangle, \langle 3, 1\rangle, \langle 2, 2\rangle\}$

Properties of the inverse and the complement

R $(\neq \emptyset)$	R^{-1}	$\overline{R}$
reflexive	reflexive	irreflexive
irreflexive	irreflexive	reflexive
symmetric	symmetric	symmetric
asymmetric	asymmetric	not symmetric
anti-symmetric	anti-symmetric	depends on R
transitive	transitive	depends on R
connected	connected	depends on R

Equivalence relations and partitions

Equivalence relations

A relation R is an **equivalence relation** iff R is reflexive, transitive and symmetric.

- examples:
  - equality
  - "has the same hair color as"
  - "is of the same age as"
  - "has the same remainder if divided by"
- notation:

$$[\![a]\!]_R := \{x | R(a, x)\}$$

- $[\![a]\!]_R$ is the set of all objects which are reachable from a via R.

Equivalence relations and partitions

Partition

Let A be a set. The set $P \subseteq \wp(A)$ is a **partition of** A iff

- $\bigcup P = A$, and
- for all $X, Y \in P$ with $X \neq Y$: $X \cap Y = \emptyset$.

- Examples: Let $A = \{a, b, c, d, e\}$
- The following sets are partitions of A:
  - $P_1 = \{\{a, c, d\}, \{b, e\}\}$
  - $P_2 = \{\{a\}, \{b\}, \{c\}, \{d\}, \{e\}\}$
  - $P_3 = \{\{a, b, c, d, e\}\}$
- The following sets are not partitions of A:
  - $C = \{\{a, b, c\}, \{b, d\}, \{e\}\}$
  - $D = \{\{a\}, \{b, e\}, \{c\}\}$

Equivalence relations and partitions

There is a tight connection between equivalence relations and partitions.

- Let $R \subseteq A \times A$ be an equivalence relation. Then the following set is a partition of $\pi_0[R]$:

  $$P_R = \{x \in \wp(A) | \text{there is a } y \in A \text{ such that } x = [\![y]\!]_R\}$$

- Let P be a partition of A. Then the following is an equivalence relation:

  $$R_P = \{\langle x, y\rangle \in A \times A | \text{there is an } X \in P \text{ with } x \in X \text{ and } y \in X\}$$

  (Instead of "$R_P(a, b)$" we sometimes write "$a \equiv_P b$".)

Equivalence relations and partitions

- example:
  - $A = \{1, 2, 3, 4, 5\}$

$$\begin{aligned} R &= \{\langle 1,1\rangle, \langle 1,3\rangle, \langle 3,1\rangle, \langle 3,3\rangle, \langle 2,2\rangle, \langle 2,4\rangle, \\ &\quad \langle 4,2\rangle, \langle 4,5\rangle, \langle 4,4\rangle, \langle 5,2\rangle, \langle 5,4\rangle, \langle 5,5\rangle, \langle 2,5\rangle\} \end{aligned}$$

  - The corresponding partition is:

$$P_R = \{\{1, 3\}, \{2, 4, 5\}\}$$

- Exercise: Let

$$\begin{aligned} R &= \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 3,3\rangle, \langle 3,5\rangle \\ &\quad \langle 5,3\rangle, \langle 5,5\rangle, \langle 4,4\rangle\} \end{aligned}$$

  What is the corresponding partition P_R?