The proof theory of partial variables

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1 Partial variables: Motivation

In Montague semantics, indefinites are interpreted as generalized quantifiers with an existential impact. Leaving aside issues of intensionality, the meaning of a phrase like (1a) comes out as (1b), being of type $\langle\langle e,t\rangle,t\rangle$.

(1) a. a man

b. $\lambda P \exists x(Mx \wedge Px)$

Post-Montagovian semanticists were not fully comfortable with this analysis. There is a common intuition that indefines are, or can be, referring expression, even if the referent is only known to the speaker, or not even to him. Formally speaking, indefinites "feel" like being of type e, despite all logical arguments to the contrary.

Discourse Representation Theory implements this intuition to some degree by assuming that indefinites introduce a variable of type e into the semantic representation of the sentence or discourse where it occurs. However, strictly speaking an indefinite like *a man* does not only introduce a free variable but an open formula, like Mx, into the representation. So technically speaking, indefinites are of type t in DRT, which is even less intuitive than the Montagovian analysis.

In a series of publications, Tanya Reinhart explored the empirical and logical consequences that result if indefinites are in fact taken to have the logical type e (Reinhart 1997; see also Winter 1997, Kratzer 1998 and much subsequent work). If one maintains the uncontroversial assumption that the head noun (plus its modifiers) in an indefinite DP expresses a property (type $\langle e,t\rangle$), the indefinite article must have type $\langle\langle e,t\rangle,e\rangle$. If one grants the rather trivial assumption that the descriptive part of an indefinite should describe the object that is referred to, this entails that the indefinite article denotes a choice function.

While this idea is intuitively appealing and led to insightful analyses of many empirical phenomena, it faces various problems, as pointed out, *inter alia*, by Reniers [1997], Geurts [2000] and Endriss [2001]. The most serious is the so-called *empty set problem.* A choice function also supplies a value if applied to the empty set. So the sentence

(2) A unicorn entered.

may be true in the actual world, depending on which choice function is used as interpretation of the indefinite article. The fundamental intuition that ordinary indefinites have an existential impact is not captured under the choice function analysis.

In Jäger [2007] I made a propsal for a compositional semantics of indefinites that preserves the fundamental assumptions of the choice function analysis but leads to Montague style truth conditions for sentences. There I make use of a new syntactic device, *partial variables.* In the next section I will briefly recapitulate this proposal.

While Jäger [2007] was couched in type theoretic terms to allow a Montague style compositional treatment, partial variables are essentially first order devices. In the present paper, I will concentrate on the extension of classical first order logic with partial variables. I will present an embedding of this logic into standard first order logic. Since there is a sound and complete proof theory for the latter, this indirectly defines a sound and complete proof theory for the former as well.

2 Semantics

The underlying idea of my proposal is inspired by DRT: Semantically, an indefinite noun phrase *is* an individual variable. So in a sentence like (2), *a unicorn* is to be translated as some variable x of type e. On the other hand, I concur with the choice function proponents. This variable must refer to some unicorn. If the local assignment function maps x to some non-unicorn, the NP does not refer.

To carry this home formally, I extend the syntax of the logical representation language with the following clause:

Definition 1 *If x is a variable of type e and φ a formula of type t, $[x|\varphi]$ is a term of type e.*

(This could be generalized to variables of arbitrary type, but in the present context type e will do.) These terms are called *partial variables.* The formula is called its *restriction.*

An indefinite like *a unicorn* would be translated as a term $[x|Ux]$. This term should behave like an ordinary variable, but it should fail to denote if x, without the restriction, would refer to some non-unicorn. So here is a first stab at the semantics of these terms:

$$\|[x|\varphi]\|_g^M = \begin{cases} g(x) \text{ if } \|\varphi\|_g^M = 1 \\ \text{undefined else} \end{cases}$$

A syntactically complex expression is assumed to be undefined (under some model and some assignment function) iff at least one of its sub-expressions is undefined. The only exceptions are the rules for the quantifiers. I only give the semantics of the existential quantifier here; the treatment of the universal quantifier is analogous.

$$\|\exists x\varphi\|_g^M = \begin{cases} 1 \text{ iff for some } a : \|\varphi\|_{g[x/a]}^M = 1 \\ 0 \text{ iff for all } a : \text{ if } \|\varphi\|_{g[x/a]}^M \text{ is defined, then } \|\varphi\|_{g[x/a]}^M = 0 \\ \text{undefined else} \end{cases}$$

The idea is that the definedness conditions of the formula in the scope of a quantifier are turned into restrictions on the truth (or falsity) condition of that quantifier. To see the motivation for this, consider the following example:

(3) a. If a (certain) linguist enters, he will be surprised.

b. $\exists x(E([x|Lx]) \to Sx)$

If the indefinite *a linguist* receives a specific interpretation, the sentence should receive the analysis given in (3b). The descriptive part of the indefinite remains *in situ*, but the variable is bound with wide scope.

The formula in the scope of the existential quantifier, $E([x|Lx]) \to Sx$ is only defined under M and g if $g(x)$ is a linguist in M. Otherwise the interpretation is standard. So (3b) comes out true if there is a linguist such that he will be surprised if he enters, and it is false otherwise. So even though the information that x is a linguist is syntactically under the scope of the implication, it is interpreted as a restriction on the existential quantifier.

This approach doesn't quite go through though. Problems arise if a formula contains more than one partial variable. In its present form, the semantics is too coarse-grained. Consider (4a), in the reading indicated in (4b).

(4) a. A yeti didn't meet a unicorn.

b. $\exists x \neg \exists y M([x|Yx], [y|Uy])$

The interpretation of the formula $M([x|Yx],[y|Uy])$ is only defined if there are both yetis and unicorns in the model. If there are no unicorns, $\exists y M([x|Yx],\ [y|Uy])$ will therefore come out false regardless of the assignment function. Therefore $\exists x \neg \exists y M([x|Yx],[y|Uy])$ will come out true even if there are no yetis.

The problem here is that the semantics does not keep the definedness conditions of different partial variables separate. To overcome this problem, I introduced a special object $\perp$ which is formally part of the universe of discourse, and which serves as dummy referent for all variables that violate their restriction. So the expression $[x|Ux]$ may have an interpretation in a model where there are no unicorns, but this interpretation must be the inconsistent object $\perp$. So the revised semantics of partial variables comes out as

Definition 2

$$\|[x|\varphi]\|_g^M = \begin{cases} g(x) \ \textit{if} \ \|\varphi\|_g^M = 1 \\ g(x) \ \textit{if} \ \|\varphi\|_g^M = 0 \ \textit{and} \ g(x) = \perp \\ \textit{undefined else.} \end{cases}$$

With this interpretation rule, $M([x|Yx],[y|Uy])$ is true under M and g if $g(x)$ is either a yeti or $\perp$ in M, and if $g(y)$ is either a unicorn or $\perp$. So even if it is known that there are no unicorns in the model, the set of assignment functions making the formula true depends on whether or not there are yetis in the model.

This information is built into the interpretation of quantfiers by demanding that they only range over consistent objects. So $\exists x \varphi$ is true if there is a value for x which is $\neq \perp$ and which makes φ true, and it is false if for all objects except $\perp$, φ is false. The formula is undefined if φ is undefined under an assignment that maps x to $\perp$. This would be the case if φ contains other partial variables except x, and their restrictions are violated.

Another problem with the initial proposal can be illustrated with the following example:

(5) a. Every girl$_i$ that met a certain boy which she$_i$ liked fainted.

b. $\exists x \forall y (Gy \wedge M(y, [x|Bx \wedge L(y,x)]) \rightarrow F(y))$

c. $\exists x (Bx \wedge \forall y (Gy \wedge Myx \wedge Lyx \rightarrow Fy))$

d. $\exists x (Bx \wedge Lyx \wedge \forall y (Gy \wedge Myx \rightarrow Fy))$

We are interested in the reading where the indefinite has scope over the entire sentence, which corresponds to the formula in (5b). The sub-formula $Gy \wedge M(y, [x|Bx \wedge L(y,x)]) \rightarrow F(y)$ is only defined under g if x is interpreted either as $\bot$ or as a boy who is loved by $g(y)$. $\forall y(Gy \wedge M(y, [x|Bx \wedge L(y,x)]) \rightarrow F(y))$ hence comes out as true if $g(x) = \bot$ and and every girl who met $\bot$ fainted, or if $g(x)$ is a boy and every girl who met and likes $g(x)$ fainted. So the entire sentence would get the intepretation indicated in (5c). The clause Lyx here ends up as restriction of the universal quantifier. Such a reading does not exist—if the indefinite has a specific reading, this clause must be part of the restriction of the existential quantifer, and the occurrence of y in this clause must be free (as indicated in (5d)). To generalize this point, we want the following equivalence to be valid regardless of free variables in ψ that appear to be bound inside φ:

$$\exists x \varphi([x|\psi]) \Leftrightarrow \exists x(\psi \wedge \varphi(x))$$

This has two consequences. First, free variable occurrences inside the restriction of a variable should not be visible from the outside while the variable that is being restricted is free. Second, as soon as a partial variable becomes bound, the free variable occurrences in it should become visible again. To take an example, in

(6) $\exists y \exists x \exists y P([x|Rxy])$

the occurrence of y inside the restriction of x should be

- free in Rxy,
- not free in $P([x|Rxy])$,
- free in $\exists x \exists y P([x|Rxy])$, and
- bound in $\exists y \exists x \exists y P([x|Rxy])$.

The intuitive idea to make this work can be explained by talking about stacks of variables. If y occurs in the restriction of x, what is seen from the outside is a stack with x on top and y underneath. Therefore y is not visible for binding. If x is bound by a quantifier though, it is taken off the stack, and y is visible again/can be bound.

Formally stacks are implemented as non-empty sequences of variables. So an assignment function is a function from V^+ to the universe (where V is the set of variables).

Here is the final version of the semantics. For the time being, I only consider the treatment of partial variables in the context of first order logic.[1]

[1] A generalization to Montague style type theory creates additional complications regarding the treatment of partial functions. Since the focus of this article is proof theory, the first order system is not just easier to handle but also more interesting because it can be axiomatized.

Definition 3 *Let $M = \langle E, F\rangle$ be a first order model (where E is the universe and F the interpretation function). $\bot \in E$ is a designated object, and $E - \{\bot\} \neq \emptyset$. g, g^x and $g\{x\}$ are assignment functions, ie. a functions from V^+ to E. $\bot$ is a designated constant. We define that for all $\vec{v} \in V^*, x, y \in V$ with $x \neq y$, and formulas φ:*

$$\begin{array}{rcl}
g^x(\vec{v}) & = & g(x\vec{v}) \\
g\{x\}(x^+) & = & g(x) \\
g\{x\}(x^*y\vec{v}) & = & g(y\vec{v}) \\
F(\bot) & = & \bot \\
\|[x|\varphi]\|_g^M & = & \begin{cases} g(x) \text{ if } \|\varphi\|_{g^x}^M = 1 \\ g(x) \text{ if } \|\varphi\|_{g^x}^M = 0 \text{ and } g(x) = \bot \\ \text{undefined else.} \end{cases} \\
\|\exists x\varphi\|_g^M & = & \begin{cases} \text{defined iff } \|\varphi\|_{g[x/\bot]\{x\}}^M \text{ is defined} \\ 1 \text{ if defined and there is an } a \in E - \{\bot\} : \|\varphi\|_{g[x/a]\{x\}}^M = 1 \\ 0 \text{ if defined and for all } a \in E - \{\bot\} : \|\varphi\|_{g[x/a]\{x\}}^M \neq 1 \end{cases} \\
\|\forall x\varphi\|_g^M & = & \begin{cases} \text{defined iff } \|\varphi\|_{g[x/\bot]\{x\}}^M \text{ is defined} \\ 1 \text{ if defined and for all } a \in E - \{\bot\} : \|\varphi\|_{g[x/a]\{x\}}^M \neq 0 \\ 0 \text{ if defined there is an } a \in E - \{\bot\} : \|\varphi\|_{g[x/a]\{x\}}^M = 0 \end{cases}
\end{array}$$

The interpretation rules for atomic formulas and the propositional connectives are standard, modulo the assumption that with the exceptions listed above, a complex expression has an interpretation under some M and g iff all its immediate subexpressions have an interpretation under M and g.

3 Syntax

In the official syntax of first order logic with partial variables, every non-empty sequence of variables is a term. Negation, implication and universal quantification are primitive connectives. The other propositional connectives and existential quantification are defined in the usual way.

Definition 4 (Syntax) *Let a set V of variables, a set C of individual constants (with $\bot \in C$) and a set $PRED_n$ of n-ary predicates be given.*

1. *Every $\vec{v} \in V^+$ is a term.*

2. *Every $c \in C$ is a term.*

3. *If $\vec{v} \in V^+$ and φ is a formula, then $[\vec{v}|\varphi]$ is a term.*

4. *If P is an n-ary predicate and $t_1, \ldots, t_n$ are terms, then $P(t_1, \ldots, t_n)$ is a formula.*

5. *If t_1, t_2 are terms, then $(t_1 = t_2)$ is a formula.*

6. *If φ and ψ are formulas, then $\neg\varphi$ and $(\varphi \to \psi)$ are formulas.*

7. *If $x \in V$ and φ is a formula, then $\forall x\varphi$ is a formula.*

8. *Nothing else is a term or a formula.*

4 Normalization

In this section I will present a normalization procedure for sets of formulas. Normal form formulas have the property that their interpretation under the semantics given above is identical to their interpretation under standard first order logic. In this way, the proof theory for FOL can be employed for our logic as well.

Normalization makes crucial use of the notion of a free occurrence of a partial variable. Due to the stack manipulations, this notion is less straightforward than in standard FOL.

Definition 5 (Free occurrence of a partial variable sequence)

1. *$[\vec{v}|\varphi]$ is a free occurrence of $\vec{v}$ in $[\vec{v}|\varphi]$.*

2. *If α is a free occurrence of $\vec{v}$ in t_i, then it is also a free occurrence of $\vec{v}$ in $P(t_1, \ldots, t_n)$ and in $t_1 = t_2$.*

3. *If α is a free occurrence of $\vec{v}$ in φ, then it is also a free occurrence of $\vec{v}$ in $\neg\varphi$, $\varphi \to \psi$ and $\psi \to \varphi$.*

4. *If α is a free occurrence of $x^n\vec{v}$ in φ and $\vec{v}_1 \neq x$, then it is a free occurrence of $\vec{v}$ in $\forall x\varphi$.*

5. *If α is a free occurrence of $\vec{v}$ in φ, then it is a free occurrence of $\vec{w}v$ in $[\vec{w}|\varphi]$.*

This notion is important because of the following lemma:

Lemma 1 *If $[\vec{u}|\varphi]$ is a free occurrence of $\vec{v}$ in A, then $\|A\|_g^M$ is defined iff $\|A(\vec{u})\|_g^M$ is defined and $\|[\vec{v}|\varphi]\|_g^M$ is defined. If defined,*

$$\|A\|_g^M = \|A(\vec{u})\|_g^M,$$

where $A(\vec{u})$ is the result of replacing $[\vec{u}|\varphi]$ in A by $\vec{u}$.

Proof: By induction over the complexity of formulas.

1. Obvious.
2. Suppose $[\vec{u}|\varphi]$ is a free occurrence of $\vec{v}$ in t_i. $P(t_1, \ldots, t_n)$ is defined under g iff all immediate subexpressions are defined under g. By induction hypothesis, this is the case iff $P(t_1, \ldots, t_i(\vec{u}), \ldots, t_n)$ is defined under g and $[\vec{v}|\varphi]$ is defined under g. If defined t_i is synonymous with $t_i(\vec{u})$ under g, and hence $P(t_1, \ldots, t_n)$ is synonymous with $P(t_1, \ldots, t_n)(\vec{v})$ under g. The argument for $t_1 = t_2$ is analogous.
3. Analogous to the previous case.
4. Suppose $[\vec{u}|\psi]$ is a free occurrence of $\vec{v}$ in φ and $\vec{v}_1 \neq x$. $\forall x\varphi$ is defined under g iff φ is defined under $g[x/\bot]\{x\}$. By induction hypothesis, this is the case iff $\varphi(\vec{u})$ and $[\vec{v}|\psi]$ are defined under $g[x/\bot]\{x\}$. As $\vec{v}_1 \neq x$, this is the case iff $\forall x\varphi(\vec{v})$ and $[\vec{v}|\psi]$ are defined under g.

   So suppose $\forall x\varphi$ is defined under g. Therefore φ is defined under $g[x/\bot]\{x\}$. By induction hypothesis, φ and $\varphi(\vec{v})$ are synonymous under $g[x/a]\{x\}$ for all $a \in E$. Therefore $\forall x\varphi$ and $(\forall x\varphi)(\vec{v})$ are synonymous under g.
5. Let $[\vec{u}|\psi]$ be a free occurrence of $\vec{v}$ in φ. Suppose $[\vec{w}|\varphi]$ is defined under g. Then φ is defined under $g^{\vec{w}}$, and either $g(\vec{w}) = \bot$ or φ is true under $g^{\vec{w}}$. By induction hypothesis, φ is defined under $g^{\vec{w}}$ iff $\varphi(\vec{u})$ and $[\vec{v}|\psi]$ are defined under $g^{\vec{w}}$. In this case, φ and $\varphi(\vec{u})$ are synonymous under $g^{\vec{w}}$. $[\vec{v}|\psi]$ is defined under $g^{\vec{w}}$ iff $[\vec{w}v|\psi]$ is defined under g. Hence $[\vec{v}|\varphi]$ is defined under g iff $[\vec{v}|\varphi](\vec{u})$ is defined under g, and hence the two expressions are synonymous under g.

$\dashv$

The normalization procedure proceeds in four steps.

4.1 Elimination of bound partial variables

This step is based on the following observation:

$$\|\forall x\varphi([\vec{v}|\psi])\|_g^M = \|\forall x(\psi \to \varphi(\vec{v}))\|_g^M, \tag{1}$$

where $[\vec{v}|\psi]$ is a free occurrence of x^n in φ.

To establish this fact, we first prove equivalence of definedness conditions. $\forall x\varphi([\vec{v}|\psi])$ is defined under M, g iff $\varphi([\vec{v}|\psi])$ is defined under $M, g[x/\bot]\{x\}$. By the previous lemma, this is the case if $\varphi(\vec{v})$ and $[x^n|\psi]$ are defined under

$M, g[x/\bot]\{x\}$. The latter is the case if ψ is defined under $g[x/\bot]\{x\}^{x^n} = g[x/\bot]\{x\}$. Likewise the right hand side is defined under M, g iff $\varphi(\vec{v}), \psi$ are defined under $g[x/\bot]\{x\}$.

Now we prove equivalence of truth conditions. Let us assume that the formulas are defined under M, g.

- Suppose
$$M, g \models \forall x \varphi([\vec{v}|\psi]).$$
Let $a \in E - \{\bot\}$. Then $\varphi([\vec{v}|\psi])$ is true or undefined under $M, g[x/a]\{x\}$. Suppose it is true. Then, according to the previous lemma, $\varphi(\vec{v})$ and ψ are true under $M, g[x/a]\{x\}$. Now suppose it is undefined. Then ψ is false under $M, g[x/a]\{x\}$. In either case, $\psi \to \varphi(\vec{v})$ is true under $M, g[x/a]\{x\}$, and hence $\forall x(\psi \to \varphi(\vec{v}))$ is true under M, g.

- Suppose
$$M, g \models \forall x(\psi \to \varphi(\vec{v})).$$
Let $a \in E - \{\bot\}$. Suppose ψ is false under $M, g[x/a]\{x\}$. Then, according to the previous lemma, $\varphi([\vec{v}|\psi)$ is undefined under $M, g[x/a]\{x\}$. Now suppose $\psi, \varphi(\vec{v})$ are true under $M, g[x/a]\{x\}$. Then $\varphi([\vec{v}|\psi)$ is true under $M, g[x/a]\{x\}$. In either case, $\forall x \varphi([\vec{v}|\psi])$ is true under M, g.

$\dashv$

Due to compositionality of interpretation, replacing a sub-formula of the form $\forall x \varphi([\vec{v}|\psi])$ within a super-formula by a synonymous formula $\forall x(\psi \to \varphi(\vec{v}))$ doesn't change interpretation. This means that all bound occurrences of partial variables in a formula can successively be removed applying the following transformation to an appropriate sub-formula:

$$\forall x \varphi([\vec{v}|\psi]) \leadsto \forall x(\psi \to \varphi(\vec{v})),$$

where $[\vec{v}|\psi]$ is a free occurrence of x^n in φ.

4.2 Elimination of redundant items on the stack

As informally described above, a quantifier $\forall x$ first removes all occurrences of x from the top of the stack (except the last one if this is the last item on the stack) before evaluating the stack. So all occurrences of x (except bottom element) on the stack in the scope of $\forall x$ are redundant. Formally, this is captured by the following equivalence:

$$\|\forall x \varphi(xy\vec{v})\|_g^M = \|\forall x \varphi(y\vec{v})\|_g^M, \tag{2}$$

(with $\vec{v} \in V^*$ and $xy\vec{v}$ free in φ) provided no quantifier $\forall y$ intervenes between $\forall x$ and the variable sequence in question. This is easily shown via induction over the complexity of φ.

This equivalence is employed in the second normalization step:

$$\forall x \varphi \rightsquigarrow \forall x' \varphi',$$

where x' is a fresh variable not occurring anywhere else in the current derivation, and φ' is the result of

- replacing all free occurrences of $x^n\vec{v}$ by $\vec{v}$ in φ, where $\vec{v} = x$ or $\vec{v}_1 \neq x$, and then
- replacing all free occurrences of x in φ by x'.

This procedure is applied inside-out, starting with the quantifiers with the narrowest scope. The renaming of variables ensures that the side condition for the equivalence is always met.

This procedure ensures that for all sub-formulas $\forall x \varphi$ of the result of this normalization step, $\|\varphi\|_{g\{x\}} = \|\varphi\|_g$. We prove this by induction over the nesting depth of quantifiers. Suppose φ does not contain any quantifiers. Since $\forall x \varphi$ does not contain bound partial variables, φ does not contain free partial occurrences of $x\vec{v}$ for any $\vec{v}$. Therefore, if $\vec{u}$ is a free occurrence of $x\vec{v}$ in φ, $\vec{u} = x\vec{v}$. But this is not possible because normalization would replace this occurrence. So the only variable sequences $\vec{v}$ with $\vec{v}_1 = x$ occurring in φ are occurrences of x. Since $g\{x\}(x) = g(x)$, the induction base follows.

So suppose that that the statement holds for all quantified sub-formulas of φ. Let $\vec{u}$ be a free occurrence of $x\vec{v}$ in φ. If $\vec{u}$ does not stand in the scope of a quantifier in φ, $\vec{u}$ is x, which can be shown with the same argument as in the previous case. So let us suppose $\vec{u}$ occurs inside a sub-formula $\forall y \psi$. The normalization procedure ensures that y is a fresh variable that does not occur in variable sequence except y inside ψ. Hence $\vec{u}$ is a free occurrence of $x\vec{v}$ in ψ. Since φ is the outcome of normalization, $\vec{v} = \varepsilon$. Then the induction step follows immediately.

4.3 Elimination of free partial variables

We need the auxiliary notion A^x, which is intended to refer to the result of prefixing x to every free occurrence of a variable sequence in the expression A. It is defined recursively as follows:

- $v^x = xv$

- $[v|\varphi]^x = [xv|\varphi]$
- $(P(t_1, \ldots, t_n))^x = P(t_1^x, \ldots, t_n^x)$
- $(\neg\varphi)^x = \neg\varphi^x$
- $(\varphi \to \psi)^x = \varphi^x \to \psi^x$
- $(\forall y\varphi)^x = \forall z[z/xy]\varphi^x$, where z is a fresh variable.

If φ is the result of the previous two normalization steps, it holds that

$$\|\varphi\|_{g^x}^M = \|\varphi^x\|_g^M. \tag{3}$$

This can easily be shown via induction over the complexity of formulas. The only non-trivial induction step concerns quantified formulas. $\|\forall y\varphi\|_{g^x} = \min_{a\in E-\{\bot\}} \|\varphi\|_{g^x[y/a]\{y\}}$. Since φ is the outcome of the previous normalization step, this equals $\min_{a\in E-\{\bot\}} \|\varphi\|_{g^x[y/a]}$, which in turn equals $\min_{a\in E-\{\bot\}} \|\varphi\|_{g[xy/a]^x} = \min_{a\in E-\{\bot\}} \|[z/xy]\varphi\|_{g[z/a]^x}$. By induction hypothesis, this equals $\min_{a\in E-\{\bot\}} \|[z/xy]\varphi^x\|_{g[z/a]} = \|(\forall y\varphi)^x\|_g$.

This equivalence is crucial for the third normalization step. This step does not preserve meaning, but truth conditions (which is not the same under a partial semantics.)

The semantic equivalence that is needed here is the following:

$$M, g \models \varphi([\vec{v}|\psi]) \Leftrightarrow M, g \models \varphi(\vec{v}) \wedge (x \neq \bot \to \psi^{\vec{v}}) \tag{4}$$

(where $A^{\vec{v}}$ is defined recursively over the elements of $\vec{v}$.) This follows directly from the definitions. Note that this is really an equivalence in truth conditions only, not in definedness conditions. If ψ is false under $g^{\vec{v}}$, then the formula on the right hand is false, but the one on the right hand side is undefined.

This equivalence of truth conditions is is used in the third normalization step:

$$\varphi([\vec{v}|\psi]) \rightsquigarrow \varphi(\vec{v}) \wedge (x \neq \bot \to \psi^{\vec{v}})$$

This is successively applied outside-in, starting with partial variables that do not occur in the restriction of another partial variable. The procedure is iterated until the formula does not contain any more partial variables.

4.4 Restricted quantification

Quantification is implicitly restricted to entities except $\bot$ in the semantics given above. This implicit restriction can be made syntactically explicit:

$$\|\forall x\varphi\|_g^M = \|\forall x(x \neq \bot \to \varphi)\|_g^M \tag{5}$$

This equivalence follows directly from the definitions.

This is employed for the final normalization step:

$$\forall x\varphi \rightsquigarrow \forall x(x \neq \bot \to \varphi),$$

which is applied in any order to all quantifiers occurring in the output of the previous normalization step.

Formulas that result from these four normalization steps do not contain partial variables; all quantifiers are explicitly restricted to objects except $\bot$, and no free variable sequence in the scope of a quantifier $\forall x$ contains redundant leading xs. Hence the interpretation of these formulas are identical under our semantics and the standard semantics for first order logic.

Normalization is in effect an embedding of the logic of partial variables into classical FOL. Since the latter has a sound and complete proof theory, it is straightforward to come up with a proof theory of the former as well. Let us say that

$$\varphi \vdash \psi$$

if and only if

$$NF(\varphi) \vdash_{FOL} NF(\psi),$$

where $NF(\varphi)$ is the result of normalizing φ, and $\vdash_{FOL}$ is some standard derivability relation in FOL, like natural deduction. Also, we say that

$$\varphi \models \psi$$

if and only if for all M, g: if $M, g \models \varphi$, then $M, g \models \psi$.

Since normalization preserves truth conditions, $\varphi \models \psi$ if and only if $NF(\varphi) \models NF(\psi)$, and since for normal form formula, interpretation under our semantics and FOL semantics coincide, $\varphi \models \psi$ if and only if $NF(\varphi) \models_{FOL} NF(\psi)$. Finally $NF(\varphi) \models_{FOL} NF(\psi)$ if and only if $NF(\varphi) \vdash_{FOL} NF(\psi)$, since the standard derivability relation in FOL is sound and complete with respect to the standard semantics. Hence $\varphi \vdash \psi$ if and only if $\varphi \models \psi$.

5 Conclusion

In the previous section I presented an embedding of the first order logic with partial variables into standard FOL. For linguistic applications, the normalization procedure can actually be simplified substantially. Logical forms of natural language sentences, to the degree that they can be translated into FOL, arguably belong to the following fragment of the logic of partial variables:

- All partial variables are bound, and
- only single variables, i.e. variable sequences of length 1 occur (free or bound).

If these conditions are met, the first normalization step boils down to repeated application of the following transformation:

$$\forall x\varphi([x|\psi]) \rightsquigarrow \forall x(\psi \rightarrow \varphi(x))$$

The second and third normalization steps are vacuous in this fragment. The last normalization step can be omitted as well if the formulas are interpreted in the domain $E - \{\bot\}$. So in this fragment, normalization essentially boils down to turning the restriction of a partial variable into a restriction of the quantifier that binds this variable.

So as far as the expressivity of the logical representation language is concerned, partial variables are syntactic sugar in first order logic; every formula with partial variables has a normal form without partial variables. It remains to be seen in future research whether this generalizes to higher order logic and λ-abstraction.

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