On relational interpretation of multi-modal categorial logics

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1 Outline of talk

- Relational interpretation of associative Lambek calculus **L**
- Moortgat's multi-modal logic **L**$\Diamond$
- Relational interpretation of **L**$\Diamond$
- The semantic effect of distributivity postulates
- Translations **L**$\Diamond \rightsquigarrow$ **L** and their semantic justification

2 Relational interpretation for L

- Associative Lambek calculus **L**:

$$A \to C/B \text{ iff } A \bullet B \to C \text{ iff } B \to A\backslash C$$

$$(A \bullet B) \bullet C \leftrightarrow A \bullet (B \bullet C)$$

- Kurtonina (1995): **L1** is sound and complete in *relational frames*:

$$\begin{aligned}
\|A \bullet B\| &= \|A\| \circ \|B\| \\
\|A\backslash B\| &= \overline{\|A\|^{\cup} \circ \overline{\|B\|}} \\
\|A/B\| &= \overline{\overline{\|A\|} \circ \|B\|^{\cup}}
\end{aligned}$$

- Pankrat'ev (1994), Andréka and Mikulás (1994): **L** is sound and complete in *relativized relational frames*

Definition 1 (Relativized relational semantics for L)

☞ For all atoms p: $V(p) \subseteq <$

$$
\begin{array}{rcl}
\langle a,b\rangle \models p & \iff & \langle a,b\rangle \in V(p) \\
\langle a,b\rangle \models A \bullet B & \iff & a < b \wedge \exists c(\langle a,c\rangle \models A \wedge \langle c,b\rangle \models B) \\
\langle a,b\rangle \models A\backslash B & \iff & a < b \wedge \forall c(\langle c,a\rangle \models A \Rightarrow \langle c,b\rangle \models B) \\
\langle a,b\rangle \models B/A & \iff & a < b \wedge \forall c(\langle b,c\rangle \models A \Rightarrow \langle a,c\rangle \models B) \\
\langle a,b\rangle \models A, X & \iff & a < b \wedge \exists c(\langle a,c\rangle \models A \wedge \langle c,b\rangle \models X)
\end{array}
$$

3 The logic L$\Diamond$

- **L**$\Diamond$: extension of **L** by pairs of unary modal operators $\Diamond_i$, $\Box^{\downarrow}{}_i$
- logical behaviour governed by residuation laws (cf. Moortgat 1996)

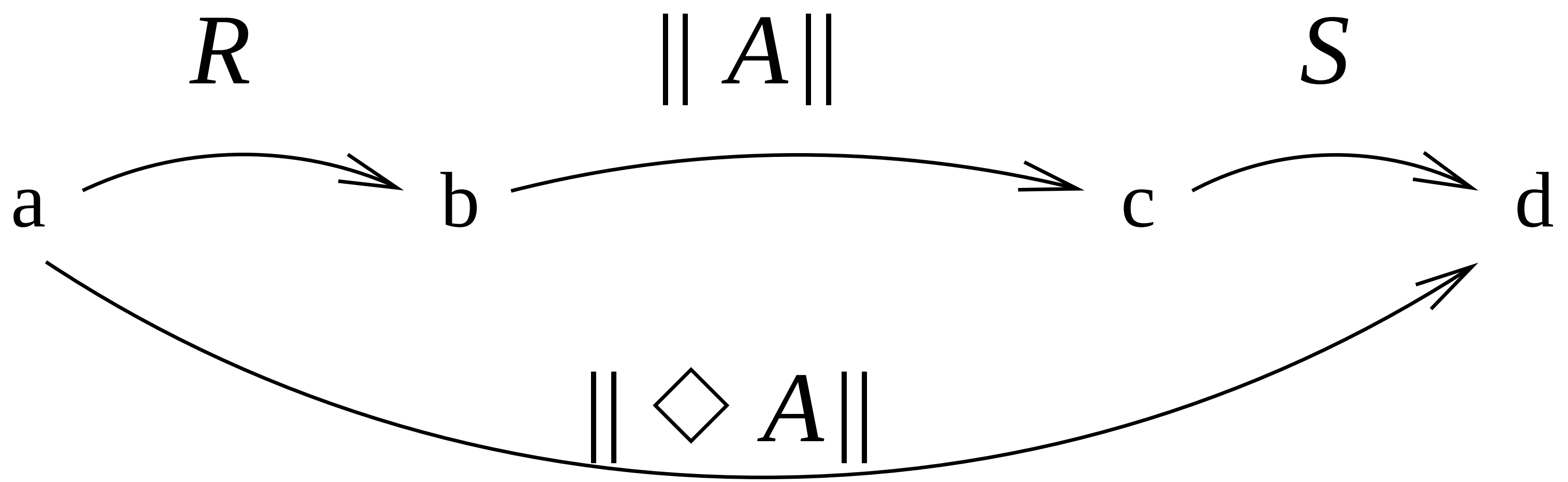
$$\Diamond A \rightarrow B \text{ iff } A \rightarrow \Box^{\downarrow} B$$

- Sequent calculus: premise sequents are bracketed strings of formulas

$$\frac{X, (A), Y \Rightarrow B}{X, \Diamond A, Y \Rightarrow B}\Diamond L \qquad \frac{X \Rightarrow A}{(X) \Rightarrow \Diamond A}\Diamond R$$

$$\frac{X, A, Y \Rightarrow B}{X, (\Box^{\downarrow} A), Y \Rightarrow B}\Box^{\downarrow} L \qquad \frac{(X) \Rightarrow A}{X \Rightarrow \Box^{\downarrow} A}\Box^{\downarrow} R$$

4 Relational interpretation of L$\Diamond$



Definition 2 (Relational Semantics for L$\Diamond$)

☞ $R, S \subseteq <$

$$
\begin{array}{rcl}
\langle a,b\rangle \models p & \iff & \langle a,b\rangle \in V(p) \\
\langle a,b\rangle \models A \bullet B & \iff & a < b \wedge \exists c(\langle a,c\rangle \models A \wedge \langle c,b\rangle \models B) \\
\langle a,b\rangle \models A\backslash B & \iff & a < b \wedge \forall c(\langle c,a\rangle \models A \Rightarrow \langle c,b\rangle \models B) \\
\langle a,b\rangle \models B/A & \iff & a < b \wedge \forall c(\langle b,c\rangle \models A \Rightarrow \langle a,c\rangle \models B) \\
\langle a,b\rangle \models \Diamond A & \iff & a < b \wedge \exists c,d(aRc \wedge \langle c,d\rangle \models A \wedge dSb) \\
\langle a,b\rangle \models \Box^{\downarrow} A & \iff & a < b \wedge \forall c,d(cRa \wedge bSd \wedge c < d \Rightarrow \langle c,d\rangle \models A) \\
\langle a,b\rangle \models A, X & \iff & a < b \wedge \exists c(\langle a,c\rangle \models A \wedge \langle c,b\rangle \models X) \\
\langle a,b\rangle \models (X) & \iff & a < b \wedge \exists c,d(aRc \wedge \langle c,d\rangle \models X \wedge dSb)
\end{array}
$$

5 Completeness of the relational semantics

Theorem 1 *For every sequent $A \Rightarrow B$:*

$$\vdash_{\mathbf{L}\Diamond} A \Rightarrow B \textit{ iff } \models A \Rightarrow B$$

- Soundness proof: Induction over derivations
- Completeness proof:
  - Labeled deduction
  - Construction of canonical model from labels

5.1 Sketch of completeness proof

Definition 3 (Labeled Sequent Calculus)

$$\frac{}{ab : A \Rightarrow ab : A}\, id$$

$$\frac{X \Rightarrow ab : A \qquad Y, ab : A, Z \Rightarrow cd : B}{Y, X, Z \Rightarrow cd : B}\, Cut$$

$$\frac{X \Rightarrow ab : A \qquad Y, ac : B, Z \Rightarrow de : C}{Y, X, bc : A \backslash B, Z \Rightarrow de : C}\, \backslash L$$

$$\frac{\underline{a}b : A, X \Rightarrow \underline{a}b : B}{X \Rightarrow bc : A \backslash B}\, \backslash R$$

$$\frac{X \Rightarrow ab : A \qquad Y, cb : B, Z \Rightarrow de : C}{Y, ca : B/A, X, Z \Rightarrow de : C}\, /L$$

$$\frac{X, a\underline{b} : A \Rightarrow c\underline{b} : B}{X \Rightarrow ca : B/A}\, /R$$

$$\frac{X, a\underline{b} : A, \underline{b}c : B, Y \Rightarrow de : C}{X, ac : A \bullet B, Y \Rightarrow de : C}\, \bullet L$$

$$\frac{X \Rightarrow ab : A \qquad Y \Rightarrow bc : B}{X, Y \Rightarrow ac : A \bullet B}\, \bullet R$$

$$\frac{X, a\underline{b} : 0, \underline{bc} : A, \underline{c}d : 1, Y \Rightarrow ef : B}{X, ad : \Diamond A, Y \Rightarrow ef : B}\, \Diamond L$$

$$\frac{X \Rightarrow ab : A}{ca : 0, X, bd : 1 \Rightarrow cd : \Diamond A}\, \Diamond R$$

$$\frac{X, cd : A, Y \Rightarrow ab : B}{X, ce : 0, ef : \Box^{\downarrow} A, fd : 1, Y \Rightarrow ab : B}\, \Box^{\downarrow} L$$

$$\frac{\underline{a}b : 0, X, c\underline{d} : 1 \Rightarrow \underline{ad} : A}{X \Rightarrow bc : \Box^{\downarrow} A}\, \Box^{\downarrow} R$$

- Conventions:
  - "0" and "1" instead of opening and closing brackets respectively
  - underlined labels do not occur elsewhere in the respective proof

5.1.1 T-F formulas

- T-F formula: labeled formula + truth value "T" or "F" (like "$Tab : p/(q\backslash r)$")
- T-F set: set of T-F formulas
- A T-F set Δ is *deeply consistent* iff
  - Δ does not contain elements $T\alpha_1 \ldots T\alpha_n, F\beta$ such that $\alpha_1 \ldots \alpha_n \Rightarrow \beta$ is derivable, and
  - Δ is *acyclic*, i.e. the transitive closure of the relation $\{\langle a, b\rangle | \exists A.Tab : A \in \Delta\}$ is irreflexive

5.1.2 Henkin witnesses

- Adding Henkin witnesses: operation on T-F sets
- serves to justify assignment of truth values to labeled formulas in T-F set Δ

$$
\begin{array}{rcl}
Tab : A \bullet B \in \Delta & \rightsquigarrow & \text{add } Tac : A, Tcb : B \ (c \text{ fresh}) \\
Fab : A \backslash B \in \Delta & \rightsquigarrow & \text{add } Tca : A, Fcb : B \ (c \text{ fresh}) \\
Fab : A / B \in \Delta & \rightsquigarrow & \text{add } Tbc : B, Fac : A \ (c \text{ fresh}) \\
Tab : \Diamond A \in \Delta & \rightsquigarrow & \text{add } Tac : 0, Tcd : A, Tdb : 1 \ (c, d \text{ fresh}) \\
Fab : \Box^{\downarrow} A \in \Delta & \rightsquigarrow & \text{add } Tca : 0, Fcd : A, Tbd : 1 \ (c, d \text{ fresh})
\end{array}
$$

☞ Adding Henkin witnesses preserves deep consistency.

5.1.3 Maximally consistent T-F set

- Maximally consistent T-F set Δ: either $Tab : A \in \Delta$ or $Fab : A \in \Delta$ for all labels a, b and formulas A.
- Every deeply consistent T-F set can be extended to a maximally consistent one:
  - If Δ is deeply consistent, then either $\Delta \cup \{Tab : A\}$ or $\Delta \cup \{Fab : A\}$ is so too
  - Starting from a d. c. set Δ, choose some enumeration of the labeled formula and add each of them with a consistency preserving truth value to Δ, followed by adding Henkin witnesses

5.1.4 Canonical model

- A maximally consistent T-F set Δ determines a canonical model M_Δ:
  - $W =$ the set of labels occurring in Δ
  - $a < b$ iff $\langle a, b\rangle \in TC(\{\langle c, d\rangle | Tcd : A \in \Delta\})$
  - $V(p) = \{\langle a, b\rangle | Tab : p \in \Delta\}$
  - $R = \{\langle a, b\rangle | Tab : 0 \in \Delta\}$
  - $S = \{\langle a, b\rangle | Tab : 1 \in \Delta\}$

$$\langle a, b\rangle \models_{M_\Delta} A \iff Tab : A \in \Delta$$

- Suppose $\mathbf{L}\Diamond \not\vdash A \Rightarrow B$
- Then $ab : A \Rightarrow ab : B$ is underivable in the labeled calculus.
- Thus $\{Tab : A, Fab : B\}$ is deeply consistent.
- By extending this set to a maximally consistent one and constructing the canonical model, we obtain a model and a pair of states that verify A and falsify B.
- Thus $\not\models A \Rightarrow B$.
- By contraposition, from $\models A \Rightarrow B$ we can infer $\mathbf{L}\Diamond \vdash A \Rightarrow B$.

$\dashv$

6 Distributivity axioms

- **L**$\Diamond$ can be enriched with interaction postulates
- Some postulates correspond to natural relational interpretations
- Example: Weak distributivity axioms

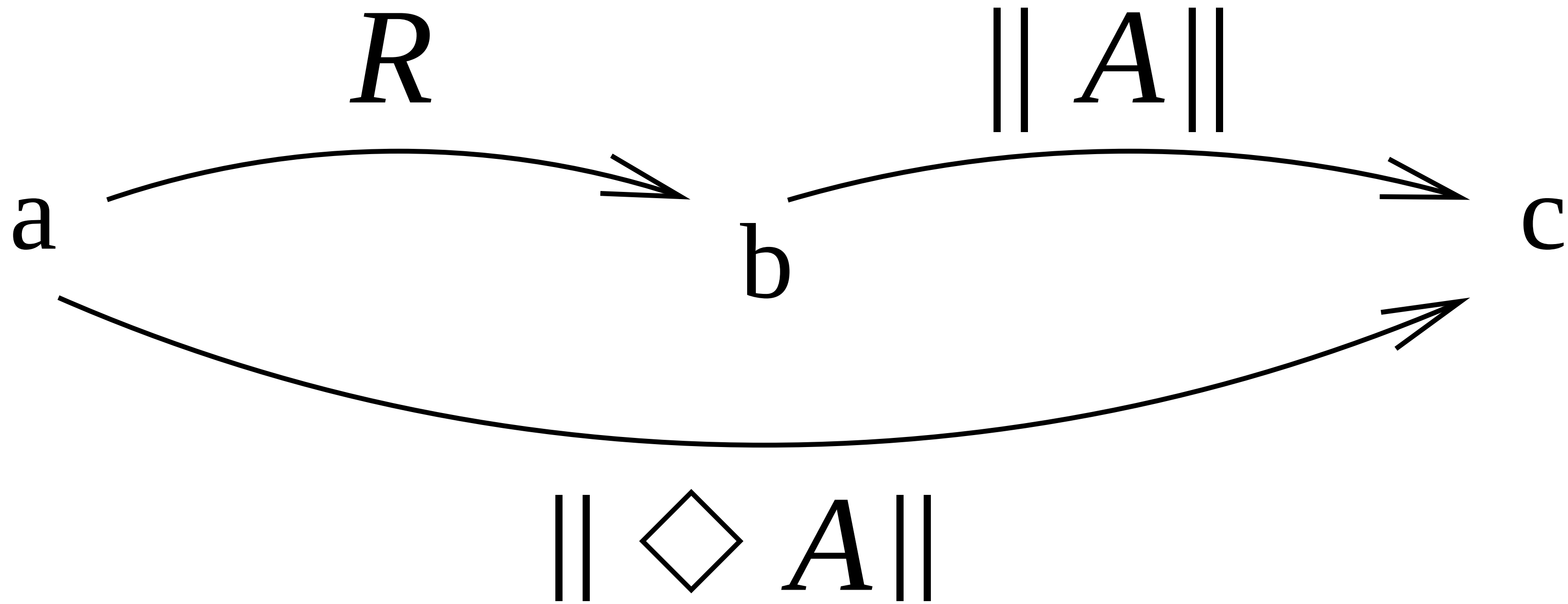
$$\Diamond A \bullet B \quad \longleftrightarrow \quad \Diamond(A \bullet B) \quad (D_l)$$
$$A \bullet \Diamond B \quad \longleftrightarrow \quad \Diamond(A \bullet B) \quad (D_r)$$

- Structural rule corresponding to D_l:

$$\frac{X, (Y), A, Z \Rightarrow B}{X, (Y, A), Z \Rightarrow B} \; D_l$$

☞ Position of closing bracket is irrelevant

7 Relational interpretation of $\mathbf{L}\diamond + D_l$



Definition 4 (Relational Semantics for L$\Diamond$+D_l)

$$\langle a,b\rangle \models \Diamond A \iff a < b \wedge \exists c(aRc \wedge \langle c,b\rangle \models A)$$

$$\langle a,b\rangle \models \Box^{\downarrow} A \iff a < b \wedge \forall c(cRa \Rightarrow \langle c,b\rangle \models A)$$

$$\langle a,b\rangle \models (X) \iff a < b \wedge \exists c(aRc \wedge \langle c,b\rangle \models X)$$

Theorem 2 *For every sequent $A \Rightarrow B$:*

$$\vdash_{\mathbf{L}\Diamond + D_l} A \Rightarrow B \text{ iff } \models A \Rightarrow B$$

Proof idea:

- Analogous to **L**$\Diamond$
- Only difference: ignore the closing brackets in the labeled calculus and thus in the model construction

8 Proof-theoretic application

- Semantic results can be used to prove validity of translations between categorial logics (cf. Kurtonina and Moortgat 1997)
- First example: Versmissens's 1996 translation $\mathbf{L}\Diamond \rightsquigarrow L$

Definition 5 (Translation $\mathbf{L}\Diamond \rightsquigarrow \mathbf{L}$)

$$\begin{array}{rcll} [p] & = & p & (p \text{ atomic}) \\ [A \bullet B] & = & [A] \bullet [B] & \\ [A \backslash B] & = & [A] \backslash [B] & \\ [A/B] & = & [A]/[B] & \\ [\Diamond A] & = & t_0 \bullet [A] \bullet t_1 & \\ [\Box^{\downarrow} A] & = & t_0 \backslash [A] / t_1 & \end{array}$$

where t_0 and t_1 are fresh atomic formulas.

Theorem 3 (Versmissen)

$$\mathbf{L}\Diamond \vdash A \Rightarrow B \text{ iff } \mathbf{L} \vdash [A] \Rightarrow [B]$$

- Versmissen gives purely syntactic proof
- *only-if* direction: induction over derivations

- *if* direction:
  - **L**$\Diamond$-model M can be transformed to **L**-model M' by having $V(t_0) = R$ and $V(t_1) = S$
  - Then
    $$\langle a, b\rangle \models_M A \text{ iff } M', \langle a, b\rangle \models_{M'} [A]$$
  - Suppose
    $$\mathbf{L}\Diamond \not\vdash A \Rightarrow B$$
  - Then by completeness of **L**$\Diamond$, there are M, a, b with
    $$\langle a, b\rangle \models_M A, \langle a, b\rangle \not\models_M B$$
  - Thus
    $$\langle a, b\rangle \models_{M'} [A], \langle a, b\rangle \not\models_{M'} [B]$$
  - By soundess of **L**:
    $$\mathbf{L} \not\vdash [A] \Rightarrow [B]$$

- Analogous construction for $\mathbf{L}\Diamond + D_l$:

Definition 6 (Translation $\mathbf{L}\Diamond + D_l \rightsquigarrow \mathbf{L}$)

$$\begin{array}{rcll} [p]^\# & = & p & (p \text{ atomic}) \\ [A \bullet B]^\# & = & [A]^\# \bullet [B]^\# & \\ [A \backslash B]^\# & = & [A]^\# \backslash [B]^\# & \\ [A/B]^\# & = & [A]^\# / [B]^\# & \\ [\Diamond A]^\# & = & t \bullet [A]^\# & \\ [\Box^\downarrow A]^\# & = & t \backslash [A]^\# & \end{array}$$

where t is a fresh atomic formula.

Theorem 4

$$\mathbf{L}\Diamond \vdash A \Rightarrow B \text{ iff } \mathbf{L} \vdash [A]^\# \Rightarrow [B]^\#$$

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