Static and dynamic stability conditions for structurally stable signaling games

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Overview

- signaling games
- costly signaling
- some examples
- conditions for evolutionary stability
- ESSets
- neutral stability
- dynamic stability and basins of attraction

Signaling games

general setup

- two players, the sender and the receiver.
- sender has private information about an event that is unknown to the receiver
- event is chosen by nature according to a certain fixed probability distribution
- sender emits a signal which is revealed to the receiver
- receiver performs an action, and the choice of action may depend on the observed signal
- utilities of sender and receiver may depend on the event, the signal and the receiver's action

Signaling games

specific assumptions

- the utility of sender and receiver are identical,
- set of events $\mathcal{E}$, set of events $\mathcal{F}$, and set of actions $\mathcal{A}$ are finite,
- $\mathcal{E} = \mathcal{A}$ (the receiver's action is to guess an event)

Signaling games

costly signaling

- production/reception of signals may incur costs
- examples:
  - length, processing complexity etc. of natural language expressions
  - advertising costs in economics
  - “handicap” signaling in biology
  - ...
- can be represented as negative utility

Signaling games

- let e be the event to be communicated, σ the signal and a the receiver's action
- c_σ is the cost of using signal σ
- partnership game: S and H have identical utility function

utility function (extensive form)

$$u(e, \sigma, a) = \delta_{e,a} + c_\sigma \tag{1}$$

Signaling games

matrix representation

- let $n = |\mathcal{E}|$ be number of events
- $m = |\mathcal{F}|$ is number of signals
- (pure) strategies can be represented as matrices with one 1 per row and else columns
- sender strategy S: $n \times m$-matrix
- receiver strategy R: $m \times n$-matrix
- $\vec{e}$: nature's probability distribution over events
- $\vec{c}$: costs of signals $1, \ldots, m$

Signaling games

normal form utility function

$$u(S,R) \quad = \quad \sum_i e_i \times \sum_j s_{ij}(r_{ji} + c_j) \qquad (2)$$

Signaling games

compiling costs and probabilities into matrix notation

$$
\begin{aligned}
p^S_{ij} &\doteq s_{ij} \times e_i \\
q^R_{ij} &\doteq r_{ij} + c_i
\end{aligned}
$$

utility function

$$
u(S,R) = \sum_i \sum_j p^S_{ij} q^R_{ji} = \mathsf{tr}(P^S Q^R).
$$

Signaling games

symmetrized mixed strategies

- let x be a mixed strategy of a symmetrized signaling game with costly signaling

$$P^x = \sum_{P,Q} x(P,Q)P \tag{3}$$

$$Q^x = \sum_{P,Q} x(P,Q)Q \tag{4}$$

Signaling games

symmetrized utility function

$$u(x,y) \quad = \quad \mathsf{tr}(P^x Q^y) + \mathsf{tr}(P^y Q^x) \tag{5}$$

Signaling games

further constraints

- costs are normalized such that $\max_i c_i = 0$
- all events have positive probability
- no event has costs ≤ -1—otherwise use of that signal would never be rationalizable

structural stability

- no two events have identical probability
- no two signals have identical costs
- all signals have costs strictly > -1

Examples

example 1: more signals than events

- $(n, m) = (2, 3)$
- $\vec{e} = \langle .6, .4 \rangle$
- $\vec{c} = \langle 0, -.1, -.4 \rangle$
- one possible Nash equilibrium:

$$P^x = \begin{pmatrix} .3 & .3 & 0 \\ .3 & 0 & .1 \end{pmatrix} \quad Q^x = \begin{pmatrix} .9 & .1 \\ .9 & -.1 \\ -.9 & .1 \end{pmatrix}$$

Examples

example 2: more events than signals

- $(n, m) = (3, 2)$
- $\vec{e} = \langle .5, .3, .2 \rangle$
- $\vec{c} = \langle 0, -.1 \rangle$
- Nash equilibrium:

$$P^x = \begin{pmatrix} .5 & 0 \\ .1 & .2 \\ 0 & .2 \end{pmatrix} \quad Q^x = \begin{pmatrix} 1 & 0 & 0 \\ -.1 & 0 & .8 \end{pmatrix}$$

Examples

example 3: a strict Nash equilibrium

- strict equlibria:
  - $n = m$
  - bijection between events and signals
  - ESSs are exactly the strict NE
- $\vec{e} = \langle .75, .25 \rangle$
- $\vec{c} = \langle 0, -.1 \rangle$

$$P^{x_1} = \begin{pmatrix} .75 & 0 \\ 0 & .25 \end{pmatrix} \quad Q^{x_1} = \begin{pmatrix} 1 & 0 \\ -.1 & .9 \end{pmatrix}$$

$$P^{x_2} = \begin{pmatrix} 0 & .75 \\ .25 & 0 \end{pmatrix} \quad Q^{x_2} = \begin{pmatrix} 0 & 1 \\ .9 & -.1 \end{pmatrix}$$

Neutral stability

Definition (Neutral stability)

The (possibly mixed) strategy profile x^* is *neutrally stable* iff

- $\forall y : u(x^*, x^*) \geq u(y, x^*)$, and
- $\forall y :$ if $u(y, x^*) = u(x^*, x^*)$, then $u(x^*, y) \geq u(y, y)$.

Examples

example 4: a neutrally stable state for the previous game

$$P^x = \begin{pmatrix} .75 & 0 \\ .25 & 0 \end{pmatrix} \quad Q^x = \begin{pmatrix} 1 & 0 \\ \alpha - .1 & .9 - \alpha \end{pmatrix}$$

for $\alpha \in (.9, 1]$.

Examples

example 5: an unstable equilibrium

$$P^x = \begin{pmatrix} .75 & 0 \\ .25 & 0 \end{pmatrix} \quad Q^x = \begin{pmatrix} 1 & 0 \\ .8 & 0 \end{pmatrix}$$

Evolutionary stability

Observation

If $n = m$, x is an ESS if and only if S^x is a permutation matrix and R^x its transpose.

Theorem

x is an ESS if and only if

1. $m \leq n$,
2. *the first column of P^x has $n - m + 1$ positive entries,*
3. *each other column of P^x has exactly one positive entry, and*
4. $q^x_{ji} = 1 + c_j$ *iff* $i = \min(\{i' : p^x_{i'j} > 0\})$, *otherwise* $q^x_{ji} = c_j$.

Evolutionary stability

an ESS with $m < n$

$$P^x = \begin{pmatrix} .5 & 0 \\ .3 & 0 \\ 0 & .2 \end{pmatrix} \quad Q^x = \begin{pmatrix} 1 & 0 & 0 \\ -.1 & -.1 & .9 \end{pmatrix}$$

Evolutionary stability

Evolutionarily stable sets

- proposed in Thomas (1985)
- generalization of ESSet
- set of Nash equilibria that is, as a whole, protected against invasions by mutants

Definition

A set A of symmetric Nash equilibria is an *evolutionarily stable set* (ESSet) if, for all $x^* \in A, u(x^*, x) > u(x, x)$ whenever $u(x, x^*) = u(x^*, x^*)$ and $x \notin A$.

Evolutionary stability

a non-singleton ESSet

$$\left\{ x: \quad P^x = \begin{pmatrix} .8 & 0 & 0 \\ 0 & .2 & 0 \end{pmatrix}, \quad Q^x = \begin{pmatrix} 1 & 0 \\ -.1 & .9 \\ \alpha - .2 & .8 - \alpha \end{pmatrix} \ \& \ \alpha \in [0,1] \right\}$$

Evolutionary stability

Theorem

A set of strategies A is an ESSet iff for each $x \in A$, x is an ESS or

1. $m > n$,
2. *the restriction of P^x to the first n columns and the restriction of Q^x to the first n rows form an ESS, and*
3. *for each y such that $P^x = P^y$, and Q^x and Q^y agree on the first n rows: $y \in A$.*

Neutral stability

Theorem

x is a NSS if and only if it is a Nash equilibrium and Q^x does not contain multiple column maxima.

Observation

If $m, n \geq 2$, there is always at least one NSS that is not element of an ESSet.

Dynamic stability

some facts

- in symmetrized asymmetric games:
  - the ESSs are exactly the asymptotically stable rest points under the replicator dynamics,
  - the ESSets are exactly the asymptotically stable sets of rest points under the replicator dynamics (Cressman, 2003)
- in doubly symmetric games,
  - the neutrally stable states are exactly the Lyapunov stable rest points (Thomas, 1985; Bomze and Weibull, 1995; Bomze, 2002)

Dynamic stability

Lemma

Let x^* be a NSS that is not an ESS. There is some $\epsilon > 0$ such that for each Nash equilibrium y with $\|x - y\| < \epsilon$,

1. *y is itself neutrally stable, and*
2. *for each $\alpha \in [0, 1]$, $\alpha x^* + (1 - \alpha)y$ is neutrally stable.*

Dynamic stability

Theorem

Each NSS x *has some non-null environment* A *such that each interior point in* A *converges to some neutrally stable equilibrium* y *under the replicator dynamics that belongs to the same continuum of NSSs as* x*.*

Dynamic stability

sketch of proof

(proof inspired by Pawlowitsch, 2006)

- suppose x is an NSS
- then x is Lyapunov stable
- for each environment U of x, every interior point in U converges to some Nash equilibrium (Hofbauer and Sigmund, 1998; Akin and Hofbauer, 1982)
- hence almost every point in some environment A of x converges to some NSS that belongs to the same continuum of NSSs as x

Dynamic stability

Corollary

The set of Nash equilibria that do not belong to any ESSet attracts a positive measure of the state space.

Dynamic stability

Theorem

Given any strategy profile x_1, there is a finite sequence of profiles $(x_i)_{i \leq n}$ for some $n \in \mathbb{N}$ such that

1. *there is an ESSet E such that $x_n \in E$, and*
2. $u(x_{i+1}, x_i) \geq u(x_i, x_i)\ \forall i < n$.

Conclusion

in a nutshell

- evolutionary stability: 1-1 map between min(m, n)-many events and signals
- if $n > m$, excess events are expressed by cheapest signal
- neutral stability: some signals may remain unused, even if they would be useful
- natural selection alone does not suffice to guarantee convergence to evolutionary stability ($=$ local maximum of average utility)
- combination of natural selection and drift does guarantee convergence to some ESSet

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